Problem Set 8

DUE: Thursday March 31 [Late papers will be accepted until 1:00 PM Friday].

1. a) Let $A$ be an $n \times n$ invertible real symmetric matrix, $b \in \mathbb{R}^n$ and $c \in \mathbb{R}$. For $x \in \mathbb{R}^n$ consider the quadratic polynomial

$$Q(x) = \langle x, Ax \rangle + \langle b, x \rangle + c.$$

Show that by a translation by some vector $v \in \mathbb{R}^n$, so $x = y + v$ in the new $y$ variable the polynomial has the form

$$Q(y) = \langle y, Ay \rangle + \gamma$$

for some real constant $\gamma$. HINT: Prove and use that for any vectors $y$ and $v$ we have $\langle Ay, v \rangle = \langle Av, y \rangle$.

[This generalizes “completing the square” from high school algebra.]

b) Let $x, y \in \mathbb{R}$. Compute

$$\iint_{\mathbb{R}^2} e^{-((2x^2 - 2xy + 3y^2) + x - 2y - 3)} dxdy$$

c) Let $h(t)$ be a given function and say you know that $\int_{-\infty}^{\infty} h(t) dt = \alpha$. If $C$ be a positive definite real (symmetric) $2 \times 2$ matrix and $x \in \mathbb{R}^2$. Show that

$$\iint_{\mathbb{R}^2} h(\langle x, Cx \rangle) dA = \frac{\pi \alpha}{\sqrt{\det C}}$$

and use this to compute

$$\iint_{\mathbb{R}^2} \frac{dxdy}{(1 + x^2 + 2xy + 5y^2)^2},$$

where $x, y \in \mathbb{R}$.

2. Let $\lambda_1$ be the lowest eigenvalue of the $n \times n$ real symmetric matrix $A$. Show that

$$\lambda_1 = \min_{x \neq 0} \frac{\langle x, Ax \rangle}{\|x\|^2}.$$

3. In class for $x \in \mathbb{R}^3$ we used the special function $v(x) = \frac{A}{|x - x_0|}$ to find a formula for a particular solution of the inhomogeneous equation $\Delta u = h(x)$:

$$u(x_0) = -\frac{1}{4\pi} \iiint_{\mathbb{R}^3} \frac{h(x)}{|x - x_0|} dx.$$

Use the same idea with $v(x) = A \log |x - x_0|$ to find a formula for a particular solution of the inhomogeneous equation $\Delta u = h(x)$ in the plane $\mathbb{R}^2$. Assume $u(x)$ (and hence $h(x)$) vanishes outside some sphere.
4. a) Let $B$ be the ball $\{r^2 = x^2 + y^2 + z^2 < a^2\}$ in $\mathbb{R}^3$. Compute all the radial eigenfunctions $u(r)$ of $-\Delta$ with Neumann boundary conditions $\partial u/\partial r = 0$ for $r = a$. Thus, you are solving $-\left[u_{rr} + \frac{2}{r}u_r\right] = \lambda u$. [SUGGESTION: the substitution $v(r) = ru(r)$ is useful. Note it implies $v(0) = 0$.]

b) Compute the corresponding eigenvalues (there is an explicit formula).

c) Use this to solve the heat equation $u_t = \Delta u$ in $B$ with $u_r = 0$ on the boundary in the special case where the initial temperature, $u(x,0) = \varphi(r)$ depends only on $r$. Your solution will be an infinite series. Please include a formula for finding the coefficients.

5. Find a bounded harmonic function in the exterior of the unit sphere $\{r > 1\}$ in $\mathbb{R}^3$ that satisfies $\partial u/\partial r = -\cos \theta$ on the boundary $r = 1$. 