Many Coupled Oscillators

A Vibrating String

Say we have \( n \) particles with the same mass \( m \) equally spaced on a string having tension \( \tau \). Let \( y_k \) denote the vertical displacement if the \( k^{th} \) mass. Assume the ends of the string are fixed; this is the same as having additional particles at the ends, but with zero displacement: \( y_0 = 0 \) and \( y_{n+1} = 0 \). Let \( \phi_k \) be the angle the segment of the string between the \( k^{th} \) and \( k+1^{st} \) particle makes with the horizontal. Then Newton’s second law of motion applied to the \( k^{th} \) mass asserts that

\[
my_k'' = \tau \sin \phi_k - \tau \sin \phi_{k-1}, \quad k = 1, \ldots, n. \tag{1}
\]

If the particles have horizontal separation \( h \), then \( \tan \phi_k = (y_{k+1} - y_k)/h \). For the case of small vibrations we assume that \( \phi_k \approx 0 \); then \( \sin \phi_k \approx \tan \phi_k = (y_{k+1} - y_k)/h \) so we can rewrite (1) as

\[
y_k'' = p^2(y_{k+1} - 2y_k + y_{k-1}), \quad k = 1, \ldots, n, \tag{2}
\]

where \( p^2 = \tau/mh \). This is a system of second order linear constant coefficient differential equations with the boundary conditions \( y_0(t) = 0 \) and \( y_{n+1}(t) = 0 \). As usual, one seeks special solutions of the form \( y_k(t) = v_k e^{\alpha t} \). Substituting this into (2) we find

\[
\alpha^2 v_k = p^2(v_{k+1} - 2v_k + v_{k-1}), \quad k = 1, \ldots, n,
\]

that is, \( \alpha^2 \) is an eigenvalue of the matrix \( p^2(T - 2I) \), where

\[
T = \begin{pmatrix}
0 & 1 & 0 & 0 & \ldots & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & \ldots & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & \ldots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \ldots & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & \ldots & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & \ldots & 0 & 1 & 0
\end{pmatrix}. \tag{3}
\]

From the work in the next section (see (9)), we conclude that

\[
\alpha_k^2 = -2p^2(1 - \cos \frac{k\pi}{n+1}) = -4p^2 \sin^2 \frac{k\pi}{2(n+1)}, \quad k = 1, \ldots, n,
\]

so

\[
\alpha_k = 2ip \sin \frac{k\pi}{2(n+1)}, \quad k = 1, \ldots, n.
\]

The corresponding eigenvectors \( V_k \) are the same as for \( T \) (see (10)). Thus the special solutions are

\[
Y_k(t) = V_k e^{2ipt \sin \frac{k\pi}{2(n+1)}}, \quad k = 1, \ldots, n,
\]

where \( Y(t) = (y_1(t), \ldots, y_n(t)) \).
**A Special Tridiagonal Matrix**

We investigate the simple \( n \times n \) real tridiagonal matrix:

\[
M = \begin{pmatrix}
\alpha & \beta & 0 & \cdots & 0 & 0 & 0 \\
\beta & \alpha & \beta & 0 & \cdots & 0 & 0 \\
0 & \beta & \alpha & \beta & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & \beta & \alpha & \beta \\
0 & 0 & 0 & \cdots & 0 & \beta & \alpha \\
\end{pmatrix} = \alpha I + \beta T,
\]

where \( T \) is defined by (3). This matrix arises in many applications, such as \( n \) coupled harmonic oscillators (see the previous section) and solving the Laplace equation numerically. Clearly \( M \) and \( T \) have the same eigenvectors and their respective eigenvalues are related by \( \mu = \alpha + \beta \lambda \). Thus, to understand \( M \) it is sufficient to work with the simpler matrix \( T \).

**Eigenvalues and Eigenvectors of \( T \)**

Usually one first finds the eigenvalues and then the eigenvectors of a matrix. For \( T \), it is a bit simpler first to find the eigenvectors. Let \( \lambda \) be an eigenvalue (necessarily real) and \( V = (v_1, v_2, \ldots, v_n) \) be a corresponding eigenvector. It will be convenient to write \( \lambda = 2c \).

Then

\[
0 = (T - \lambda I)V = \begin{pmatrix}
-2c & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\
1 & -2c & 1 & 0 & \cdots & 0 & 0 & 0 \\
0 & 1 & -2c & 1 & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \cdots & -2c & 1 & 0 \\
0 & 0 & 0 & 0 & \cdots & 1 & -2c & 1 \\
0 & 0 & 0 & 0 & \cdots & 0 & 1 & -2c \\
\end{pmatrix} \begin{pmatrix}
v_1 \\
v_2 \\
v_3 \\
\vdots \\
v_{n-2} \\
v_{n-1} \\
v_n \\
\end{pmatrix} = \begin{pmatrix}
v_1 - 2cv + v_2 \\
v_2 - 2cv + v_3 \\
\vdots \\
v_{k-1} - 2cv_k + v_{k+1} \\
\vdots \\
v_{n-2} - 2cv_{n-1} + v_n \\
v_{n-1} - 2cv_n \\
\end{pmatrix}
\]

(4)

Except for the first and last equation, these have the form

\[
v_{k-1} - 2cv_k + v_{k+1} = 0.
\]

(5)
We can also bring the first and last equations into this same form by introducing new artificial variables \( v_0 \) and \( v_{n+1} \), setting their values as zero: \( v_0 = 0 \), \( v_{n+1} = 0 \).

The result (5) is a second order linear difference equation with constant coefficients along with the boundary conditions \( v_0 = 0 \), and \( v_{n+1} = 0 \). As usual for such equations one seeks a solution with the form \( v_k = r^k \). Equation (5) then gives \( 1 - 2cr + r^2 = 0 \) whose roots are

\[ r_\pm = c \pm \sqrt{c^2 - 1} \]

Note also \( 2c = r + r^{-1} \) and \( r_+ r_- = 1 \). (6)

**Case 1:** \( c \neq \pm 1 \). In this case the two roots \( r_\pm \) are distinct. Let \( r := r_+ = c + \sqrt{c^2 - 1} \). Since \( r_- = c - \sqrt{c^2 - 1} = 1/r \), we deduce that the general solution of (4) is

\[ v_k = Ar^k + Br^{-k}, \quad k = 2, \ldots, n-1 \] (7)

for some constants \( A \) and \( B \) which.

The first boundary condition, \( v_0 = 0 \), gives \( A + B = 0 \), so

\[ v_k = A(r^k - r^{-k}), \quad k = 1, \ldots, n-1. \] (8)

Since for a non-trivial solution we need \( A \neq 0 \), the second boundary condition, \( v_{n+1} = 0 \), implies

\[ r^{n+1} - r^{-(n+1)} = 0, \quad \text{so} \quad r^{2(n+1)} = 1. \]

In particular, \( |r| = 1 \). Using (6), this gives \( 2|c| \leq |r| + |r|^{-1} = 2 \). Thus \( |c| \leq 1 \). In fact, \( |c| < 1 \) because we are assuming that \( c \neq \pm 1 \).

**Case 2:** \( c = \pm 1 \). Then \( r = c \) and the general solution of (4) is now

\[ v_k = (A + Bk)c^k. \]

The boundary condition \( v_0 = 0 \) implies that \( A = 0 \). The other boundary condition then gives \( 0 = v_{n+1} = B(n+1)c^{n+1} \). This is satisfied only in the trivial case \( B = 0 \). Consequently the equations (4) have no non-trivial solution for \( c = \pm 1 \).

It remains to rewrite our results in a simpler way. We are in Case 1 so \( |r| = 1 \). Thus \( r = e^{i\theta} \), \( c = \cos \theta \), and \( 1 = r^{2(n+1)} = e^{2i(n+1)\theta} \). Consequently \( 2(n+1)\theta = 2k\pi \) for some \( 1 \leq k \leq n \) (we exclude \( k = 0 \) and \( k = n+1 \) because we know that \( c \neq \pm 1 \), so \( r \neq \pm 1 \)).

Normalizing the eigenvectors \( V \) by the choice \( A = 1/2i \), we summarize as follows:
Theorem 1 The $n \times n$ matrix $T$ has the eigenvalues
\[ \lambda_k = 2c = 2\cos \theta = 2\cos \frac{k\pi}{n+1}, \quad 1 \leq k \leq n \] (9)
and corresponding eigenvectors
\[ V_k = (\sin \frac{k\pi}{n+1}, \sin \frac{2k\pi}{n+1}, \ldots, \sin \frac{n \pi}{n+1}). \] (10)

Remark 1. If $n = 2k+1$ is odd, then the middle eigenvalue is zero because $(k+1)\pi/(n+1) = (k+1)\pi/2(k+1) = \pi/2$.

Remark 2. Since $2ab = a^2 + b^2 - (a-b)^2 \leq a^2 + b^2$ with equality only if $a = b$, we see that for any $x \in \mathbb{R}^n$
\[ \langle x, Tx \rangle = 2(x_1x_2 + x_2x_3 + \cdots + x_{n-1}x_n) \leq x_1^2 + 2(x_2^2 + \cdots + x_{n-1}^2) + x_n^2 \leq 2\|x\|^2 \]
with equality only if $x = 0$. Similarly $\langle x, Tx \rangle \geq -2\|x\|^2$. Thus, the eigenvalues of $T$ are in the interval $-2 < \lambda < 2$. Although we obtained more precise information above, it is useful to observe that we could have deduced this so easily.

Remark 3. Gershgorin’s circle theorem is also a simple way to get information about the eigenvalues of a square (complex) matrix $A = (a_{ij})$. Let $D_i$ be the disk whose center is at $a_{ii}$ and radius is $R_i = \sum_{j \neq i} |a_{ij}|$, so
\[ |\lambda - a_{jj}| \leq R_j. \]
These are the Gershgorin disks.

Theorem 2 (Gershgorin) Each eigenvalues of $A$ lies in at least one of these Gershgorin discs.

Proof: Say $Ax = \lambda x$ and say $|x_i| = \max_j |x_j|$. The $i^{th}$ component of $Ax = \lambda x$ is
\[ (\lambda - a_{ii})x_i = \sum_{j \neq i} a_{ij}x_j \]
so
\[ |(\lambda - a_{ii})x_i| \leq \sum_{j \neq i} |a_{ij}||x_j| \leq R_i|x_i|. \]
That is, $|\lambda - a_{ii}| \leq R_i$, as claimed.

By Gershgorin’s theorem, we observed immediately that all of the eigenvalues of $T$ satisfy $|\lambda| \leq 2$. 


DETERMINANT OF $T - \lambda I$

We use recursion on $n$, the size of the $n \times n$ matrix $T$. It will be convenient to build on (4) and let $D_n = \det(T - \lambda I)$. As before, let $\lambda = 2c$. Then, expanding by minors using the first column of (4) we obtain the formula

$$D_n = -2cD_{n-1} - D_{n-2} \quad n = 3, 4, \ldots .$$

(11)

Since $D_1 = -2c$ and $D_2 = 4c^2 - 1$, we can use (11) to define $D_0 := 1$. The relation (11) is, except for the sign of $c$, is identical to (5). The solution for $c \neq \pm 1$ is thus

$$D_k = A s^k + B s^{-k}, \quad k = 0, 1, \ldots ,$$

(12)

where

$$-2c = s + s^{-1} \quad \text{and} \quad s = -c + \sqrt{c^2 - 1}.$$  

(13)

This time we determine the constants $A, B$ from the initial conditions $D_0 = 1$ and $D_1 = -2c$. The result is

$$D_k = \begin{cases} 
\frac{1}{2\sqrt{c^2 - 1}} (s^{k+1} - s^{-(k+1)}) & \text{if } c \neq \pm 1, \\
(-c)^k (k + 1) & \text{if } c = \pm 1. 
\end{cases}$$

(14)

For many purposes it is useful to rewrite this.

**Case 1:** $|c| < 1$. Then $s = -c + i\sqrt{1 - c^2}$ has $|s| = 1$ so $s = e^{i\alpha}$ and $c = -\cos \alpha$ for some $0 < \alpha < \pi$. Therefore from (14),

$$D_k = \frac{\sin(k + 1)\alpha}{\sin \alpha}.$$  

(15)

**Case 2:** $c > 1$. Write $c = \cosh \beta$ for some $\beta > 0$. Since $-e^\beta - e^{-\beta} = -2c = s + s^{-1}$, write $s = -e^\beta$. Then from (14),

$$D_k = (-1)^k \frac{\sinh(k + 1)\beta}{\sinh \beta},$$

(16)

where we chose the sign in $\sqrt{c^2 - 1} = -\sinh \beta$ so that $D_0 = 1$.

**Case 3:** $c < -1$. Write $c = -\cosh \beta$ for some $\beta > 0$. Since $e^t + e^{-t} = -2c = s + s^{-1}$, write $s = e^\beta$. Then from (14),

$$D_k = \frac{\sinh(k + 1)\beta}{\sinh \beta},$$

(17)

where we chose the sign in $\sqrt{c^2 - 1} = +\sinh t$ so that $D_0 = 1$.

Note that as $t \to 0$ in (15)–(17), that is, as $c \to \pm 1$. these formulas agree with the case $c = \pm 1$ in (14).