Math 425 | **Final Exam** | Jerry L. Kazdan
May 7, 2015 | | 9:00 – 11:00

DIRECTIONS This exam has two parts. Part A has 5 shorter problems (8 points each, so 40 points), while Part B has 5 traditional problems (15 points each, so total is 75 points). Maximum score is thus $40 + 75 = 115$ points.

Closed book, no calculators or computers – but you may use one sheet of $8.5'' \times 11''$ paper with notes on ONE side.

Please remember to silence your cellphone before the exam and keep it out of sight for the duration of the test period. This exam begins promptly at 9:00 and ends at 11:00; anyone who continues working after time is called may be denied the right to submit his or her exam or may be subject to other grading penalties. Please indicate what work you wish to be graded and what is scratch. *Clarity and neatness count.*

Part A 5 shorter problems, 8 points each.

A–1. Find a function $u(x,t)$ that satisfies $u_t - u = 7x$ with $u(x,0) = 0$.

SOLUTION: A particular solution of the inhomogeneous equation is $u_p = -7x$. The general solution of the homogeneous equation is $u_0 = k(x)e^t$ for any function $k(x)$. Thus the general solution of the inhomogeneous equation is

$$u(x,t) = k(x)e^t - 7x.$$

To satisfy the initial condition we need

$$0 = u(x,0) = k(x) - 7x$$

so $k(x) = 7x$ and $u(x,t) = 7xe^t - 7x$.

A–2. Let $\Omega \subset \mathbb{R}^2$ be a bounded region with boundary $\partial\Omega$.

Say $u(x,y,t)$ is a solution of $u_t - \Delta u - 2u = e^t \sin(x+2y)$ in Ω with
$u(x,y,t) = 0$ on $\partial\Omega$ and $u(x,y,0) = 0$;

and $v(x,y,t)$ is a solution of $v_t - \Delta v - 2v = 0$ in Ω with
$v(x,y,t) = x^2 y$ on $\partial\Omega$ and $w(x,y,0) = \cos(2x)$.

Find a function $w(x,y,t)$ that satisfies $w_t = \Delta w + 2w + 3e^t \sin(x+2y)$ in Ω with
$w(x,y,t) = 5x^2 y$ on $\partial\Omega$ and $w(x,y,0) = 5\cos(2x)$.

[Your solution should give a simple formula for w in terms of u and v].

SOLUTION: By linearity, $w(x,y,t) = 3u(x,y,t) + 5v(x,y,t)$.

A–3. Say $u(x,t)$ is a solution of the wave equation $u_{tt} = 9u_{xx}$ for all $-\infty < x < \infty$, with $u(x,0) = f(x)$ and $u_t(x,0) = g(x)$.

In the (x,t)-plane, find all the points on the x-axis that can influence the solution at $x = 2$, $t = 4$.

SOLUTION: The d'Alembert formula gives

$$u(x,t) = \tfrac{1}{2}[f(x+3t) + f(x-3t)] + \tfrac{1}{2\cdot 3}\int_{x-3t}^{x+3t} g(s)\,ds.$$

Letting $x = 2$ and $t = 4$ we find that the only points on the x-axis that can influence the solution are in the interval $-10 \le x \le 14$.

A–4. Let Ω be the unit disk in the plane $\mathbb{R}^2$. Let $-\Delta v_1 = \lambda_1 v_1$, where λ_1 is the lowest eigenvalue of the Laplacian in Ω and $v_1(x,y)$ is the corresponding eigenfunction with $v_1(x,y) = 0$ on $\partial\Omega$.

Use the inscribed and circumscribed squares $\Omega_\pm$ for the disk to find numbers α and β so that $0 < \alpha < \lambda_1(\Omega) < \beta$.

SOLUTION: From the Rayleigh-Ritz quotient, if $\lambda_1(\Omega)$ is the lowest eigenvalue of the Laplacian with zero Dirichlet boundary condition and if $\Omega \subset \Omega_+$, then $\lambda_1(\Omega_+) < \lambda_1(\Omega)$.

Apply this where Ω_+ is the square with corners at $(\pm 1, \pm 1)$ and $(\pm 1, \mp 1)$. Recall (or compute!) that for a rectangle in the plane with sides a and b then the eigenvalues of the Laplacian are

$$\lambda_{k,\ell} = \left(\frac{k\pi}{a}\right)^2 + \left(\frac{\ell\pi}{b}\right)^2, \qquad \ell = 1, 2, 3, \ldots$$

so for our circumscribed square $(a = b = 2)$, $\lambda_1(\Omega) > \lambda_1(\Omega_+) = 2\frac{\pi^2}{4}$.

Similarly, for the inscribes square Ω_- (so $a = b = \sqrt{2}$), $\lambda_1(\Omega) < \lambda_1(\Omega_-) = 2\frac{\pi^2}{2}$.

A–5. Let $\Omega \subset \mathbb{R}^3$ be a bounded region with smooth boundary $\partial\Omega$. Let u and v be harmonic functions in Ω with $u = f$ on $\partial\Omega$ and $v = g$ on $\partial\Omega$. If $f \ge g$, show that $u \ge v$ in Ω.

SOLUTION: Let $w = u - v$. Then $\Delta w = \Delta u - \Delta v = 0$ while $w = f - g \ge 0$ on the boundary. By the maximum principle, $w \ge 0$ in Ω. Thus $u \ge v$ in Ω.

Part B 5 standard problems (15 points each, so 75 points)

B–1. Suppose $f(x) = \begin{cases} -1 & \text{for } -\pi \le x \le 0 \\ 1 & \text{for } 0 < x \le \pi \end{cases}.$

a) Compute the Fourier series of $f(x)$ for the interval $-\pi \le x \le \pi$.

SOLUTION: The Fourier series is

$$f(x) \sim a_0 + \sum_{k=1}^{\infty} \left[a_k \cos kx + b_k \sin kx \right], \tag{1}$$

where

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\, dx, \quad a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos kx\, dx, \quad b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin kx\, dx.$$

Because f is odd, the $a_k = 0$. Also,

$$b_k = \frac{2}{\pi} \int_0^{\pi} \sin kx\, dx = -\frac{2}{k\pi} \cos kx \Big|_0^{\pi} = \begin{cases} 0 & k \text{ even} \\ \frac{4}{k\pi} & k \text{ odd} \end{cases}$$

Thus,

$$f(x) \sim \frac{4}{\pi} \left[\sin x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \frac{\sin 7x}{7} + \cdots \right].$$

b) Draw a graph of the Fourier series (computed above) for $-2\pi \le x \le 2\pi$ and put an "X" at all points of discontinuity.

SOLUTION: `https://upload.wikimedia.org/wikipedia/commons/thumb/2/2c/Fourier_Series.svg/`
There are jump discontinuities at $x = k\pi,\ k = 0, \pm 1, \pm 2, \ldots$.

c) Give a formula relating the Fourier coefficients to $\int_{-\pi}^{\pi} f^2(x)\, dx$.

SOLUTION: Taking the inner product of both sides of the formula (1) with itself gives the Parseval formula (a generalization of the Pythagorean Theorem)

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} f^2(x)\, dx = a_0^2 + \frac{1}{2} \sum_{k=1}^{\infty} \left[a_k^2 + b_k^2 \right]$$

For this example it results in the hardly obvious formula

$$\frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots$$

B–2. Use separation of variables to solve the wave equation $u_{tt} = u_{xx}$ for $0 \le x \le \pi$ with

boundary conditions: $u(0,t) = 0$ and $u_x(\pi, t) = 0$,

and

initial conditions: $u(x,0) = \sin(3x/2) - 7\sin(5x/2)$ and $u_t(x,0) = 0$.

SOLUTION: We seek special solutions of this wave equation of the form $u(x,t) = v(x)T(t)$ where $v(0) = 0$ and $v_x(\pi) = 0$. This gives

$$\frac{\ddot{T}(t)}{T} = \frac{v''(x)}{v} = -\lambda,$$

where, using the boundary conditions on v in the Rayleigh quotient, we find that the constant $\lambda > 0$. Thus,

$$\ddot{T}(t) + \lambda T = 0 \qquad \text{and} \qquad v''(x) + \lambda v = 0.$$

The solution of the equation for v gives

$$v(x) = C \cos \sqrt{\lambda} x + D \sin \sqrt{\lambda}\, x = 0,$$

where C and D are constants. The boundary condition $v(0) = 0$ shows that $C = 0$ while the boundary condition $v_x(\pi) = 0$ implies that $D\sqrt{\lambda}\, \cos(\sqrt{\lambda}\pi) = 0$. This second boundary condition shows that $\sqrt{\lambda}\pi$ must be an odd multiple of $\pi/2$, that is, $\sqrt{\lambda_k} = k/2$ for some *odd* integer k (so $\lambda_k = k^2/4$).

Using this we can solve the ODE for $T_k(t)$:

$$T_k(t) = A_k \cos(kt/2) + B_k \sin(kt/2) \qquad (k \text{ odd})$$

and thus obtain the *standing wave solutions*

$$u_k(x,t) = [A_k \cos(kt/2) + B_k \sin(kt/2)] \sin(kx/2) \qquad (k \text{ odd})$$

The general solution of the wave equation with these boundary conditions is thus

$$u(x,t) = \sum_{k \text{ odd}} [A_k \cos(kt/2) + B_k \sin(kt/2)] \sin(kx/2).$$

We now choose the coefficients A_k and B_k to satisfy the initial conditions:

$$\sin(3x/2) - 7\sin(5x/2) = u(x,0) = \sum_{k \text{ odd}} A_k \sin(kx/2) \tag{2}$$

and

$$0 = u_t(x,0) = \sum_{k \text{ odd}} B_k (k/2) \sin(kx/2). \tag{3}$$

Initial condition (2) implies that $A_3 = 1$, $A_5 = -7$ and the other $A_k = 0$ while initial condition (3) implies that all the $B_k = 0$. Consequently,

$$u(x,t) = \cos(3t/2)\sin(3x/2) - 7\cos(5t/2)\sin(5x/2).$$

B–3. For $-\infty < x < \infty$ and $t > 0$ consider the diffusion equation

$$u_t = u_{xx} + 2u_x + u \quad \text{with} \quad \mathrm{u}(x,0) = e^{-x^2} \tag{4}$$

a) Show that by making the change of variables $u(x,t) = e^{ax+bt}v(x,t)$ using a clever choice of the constants a and b, the function v satisfies the standard diffusion equation

$$v_t = v_{xx}$$

but with a modified initial condition, $v(x,0)$.

SOLUTION: We compute:

$$u_t = e^{ax+bt}v_t + be^{ax+bt}v, \qquad u_x = e^{ax+bt}v_x + ae^{ax+bt}v,$$

$$u_{xx} = e^{ax+bt}v_{xx} + 2ae^{ax+bt}v_x + a^2e^{ax+bt}v.$$

In these variables, after canceling the common e^{ax+bt} factors the equation (4) becomes

$$v_t + bv = v_{xx} + 2av_x + a^2v + 2(v_x + av) + v,$$

that is,

$$v_t = v_{xx} + (2a+2)v_x + (a^2 + 2a + 1 - b)v.$$

To eliminate the v_x term let $a = -1$. Then eliminate the v term by choosing $b = 0$. Thus, the resulting change of variables is $u(x,t) = e^{-x}v(x,t)$. The resulting equation for v is, as desired,

$$v_t = v_{xx}$$

with initial condition

$$v(x,0) = e^x u(x,0) = e^{x-x^2}.$$

b) Use this to write a formula (involving an integral) for the solution of equation (4) with the specified initial condition.

SOLUTION: Using the standard formula for the solution of the heat equation with given initial value, we obtain

$$v(x,t) = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{\frac{(x-y)^2}{4t}} e^{y-y^2}\, dy$$

Therefore

$$u(x,t) = \frac{e^{-x}}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{\frac{(x-y)^2}{4t}} e^{y-y^2}\, dy$$

REMARK: In this problem we tried the substitution $u(x,t) = e^{ax+bt}v(x,t)$ and were "lucky" that we could find constants a and b so that it was useful. Instead of relying on luck (or intuition), we could have tried the more general substitution $u(x,t) = \varphi(x,t)v(x,t)$ and, without being smart, been led to this. This more general approach also is effective in some cases where the coefficients of u_x and u in the original equation are not necessarily constants. This is a valuable exercise.

B–4. For $(x, y, z) \in \mathbb{R}^3$, let $u(x, y, z, t)$ be a solution of the Klein-Gordon equation

$$u_{tt} - \Delta u + u = 0.$$

Let

$$E(t) = \tfrac{1}{2} \iiint_{\mathbb{R}^3} \left[u_t^2 + |\nabla u|^2 + u^2 \right] \, d\,\mathrm{Vol}.$$

a) Assuming $|u(x, y, t)|$ is small for $R = \sqrt{x^2 + y^2 + z^2} \to \infty$, show that $E(t)$ is a constant.

SOLUTION:

$$\frac{dE}{dt} = \iiint_{\mathbb{R}^3} [u_t u_{tt} + \nabla u \cdot \nabla u_t + u u_t] \, d\,\mathrm{Vol}.$$

On the boundary of the ball o B_R of radius R centered at the origin, in spherical coordinates the outer normal derivative $\nabla u \cdot N = u_r$. By Green's Theorem

$$\iiint_{B_R} \nabla u \cdot \nabla u_t \, d\,\mathrm{Vol} = \iint_{x^2+y^2+z^2=R^2} u_t u_r \, dS - \iint_{B_R} u_t \Delta u \, d\,\mathrm{Vol}.$$

As $R \to \infty$, since we assumed that the solution decays for R large, the integral on the large sphere tends to zero. Thus

$$\frac{dE}{dt} = \iiint_{\mathbb{R}^3} u_t [u_{tt} - \Delta u + u] \, d\,\mathrm{Vol} = 0.$$

so $E(t)$ is a constant.

b) Use this to state and prove a uniqueness result for the solution of the Klein-Gordon equation with specified initial position and velocity.

SOLUTION: Say u and v are both solutions of this partial differential equation with the *same* initial conditions

$$u(x, y, z, 0) = v(x, y, z, 0) \quad \text{and} \quad u_t(x, y, z, 0) = v_t(x, y, z, 0)$$

and also decaying for R large. Let $w = u - v$. Then w satisfies $w_{tt} - \Delta w + w = 0$ in Ω, $w(x, y, z, 0) = 0$, $w_t(x, y, z, 0)$ and w decays for R large. By part a), the energy, $E(t)$, associated with w is a constant. Using the initial conditions, this constant is zero. Because $E(t)$ is a sum of non-negative terms, $w(x, y, z, t) = 0$. Consequently $u = v$.

B–5. Let Ω be a bounded set in $\mathbb{R}^3$ with smooth boundary, let $f(\vec{x})$ be a smooth function on Ω and let $g(\vec{x})$ be a smooth function on $\partial\Omega$. Define

$$J(w) = \tfrac{1}{2} \iiint_{\Omega} \left[|\nabla w|^2 + 2fw \right] \, d\,\mathrm{Vol}.$$

If a smooth function $u(\vec{x})$ minimizes J among all smooth functions $w(\vec{x})$ for which $w = g$ on $\partial\Omega$, show that

$$\Delta u = f \quad \text{in} \quad \Omega \quad \text{and} \quad u = g \text{ on } \quad \partial\Omega.$$

SOLUTION: Let $h(\vec{x})$ be any smooth function that is zero on $\partial\Omega$. Then for real t the function $u(\vec{x}) + th(\vec{x}) = g(\vec{x})$ on $\partial\Omega$. By the assumed minimizing property of u we have

$$J(u) \le J(u + th).$$

Therefore the function $\varphi(t) := J(u + th)$ has a minimum at $t = 0$. By basic calculus, $\varphi'(0) = 0$. However,

$$\varphi(t) = J(u + th) = \frac{1}{2}\iint_\Omega \Big[|\nabla u|^2 + 2t\nabla u \cdot \nabla h + t^2|\nabla h|^2 + 2fu + 2tfh\Big] \, \mathrm{d}\,\mathrm{Vol}$$

Therefore

$$0 = \varphi'(0) = \iiint_\Omega \nabla u \cdot \nabla h + fh \, d\,\mathrm{Vol}.$$

By Green's Theorem, since $h = 0$ on the boundary, this implies that

$$\iiint_\Omega \big(-\Delta u + f\big) h \, d\,\mathrm{Vol} = 0. \tag{5}$$

Note that this holds for *all* smooth functions h that are zero on the boundary of Ω.

To show that $\Delta u = f$ throughout Ω, say $\Delta u - f$ is positive at some point p in Ω. Then by continuity it is positive in a small ball B centered at p. Pick a smooth function h that is positive in this small ball and zero outside of it. Then the integral in (5) is positive, a contradiction.

There is a similar contradiction if $\Delta u - f$ is negative somewhere. Therefore $\Delta u = f$ throughout Ω.