Lorentz Transformations

Orthogonal Transformations

In Euclidean space \( \mathbb{R}^2 \) (and \( \mathbb{R}^n \)), it is valuable to find all the invertible linear changes of variable \( y = Rx \) that preserve the length of a vector

\[ \| Rx \| = \| x \| \]

so that

\[ y_1^2 + y_2^2 = x_1^2 + x_2^2. \]

These are orthogonal transformations. Say

\[ x_1 = ay_1 + by_2 \]
\[ x_2 = cy_1 + dy_2. \]

Then

\[ x_1^2 + x_2^2 = (a^2 + c^2)y_1^2 + 2(ab + cd)y_1y_2 + (b^2 + d^2)y_2^2. \]

We therefore want

\[ a^2 + c^2 = 1, \quad ab + cd = 0, \quad \text{and} \quad b^2 + d^2 = 1 \]

There are four variables and only three conditions so we will have one free parameter. To satisfy the first condition it is natural to let \( a = \cos \theta \) and \( c = \sin \theta \). For the second condition, let \( b = -c = -\sin \theta \) and \( d = a = \cos \theta \). The third condition is also satisfied. This gives the matrix

\[ R = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \]

These are the rotations of the plane \( \mathbb{R}^2 \).
By a similar computation, these are also the only linear changes of variable that preserve the Laplace operator

\[ \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} = \frac{\partial^2 u}{\partial y_1^2} + \frac{\partial^2 u}{\partial y_2^2}. \]

**Lorentz Transformations**

It is also valuable to find all linear changes of variable

\[ x' = \alpha x + \beta t \]
\[ t' = \gamma x + \delta t \]

that preserve the wave equation

\[ \frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t'^2} - c^2 \frac{\partial^2 u}{\partial x'^2}, \]

where \( c \) is a constant (the speed of sound or light).

By the chain rule,

\[ u_{tt} - c^2 u_{xx} = (\delta^2 - c^2 \gamma^2) u_{tt'} + 2(\beta \delta - c^2 \alpha \gamma) u_{tx'} + (\beta^2 - c^2 \alpha^2) u_{xx'}. \]

Thus we want

\[ \delta^2 - c^2 \gamma^2 = 1, \quad \beta \delta - c^2 \alpha \gamma = 0, \quad \text{and} \quad \beta^2 - c^2 \alpha^2 = -c^2 \]

First pick \( \gamma \) and \( \delta \) so that \( \delta^2 - c^2 \gamma^2 = 1 \), and then let \( \beta = \pm c^2 \gamma \), \( \alpha = \pm \delta \). To preserve orientation we use the + signs. Since \( c^2 \alpha^2 - \beta^2 = c^2 \) and \( \cosh^2 \sigma - \sinh^2 \sigma = 1 \), it is traditional to write \( \alpha = \cosh \sigma \), \( \beta = c \sinh \sigma \). For any real \( \sigma \) the transformation

\[ x' = (\cosh \sigma) x + (c \sinh \sigma) t \]
\[ t' = \left( \frac{1}{c} \sinh \sigma \right) x + (\cosh \sigma) t \] (1)
preserves the wave operator. This is called a Lorentz transformation. Lorentz [1853–1928] transformations also preserve arc length $ds^2 := dx'^2 - c^2 dt'^2 = dx^2 - c^2 dt^2$ in space-time and are fundamental in the study of the wave operator and special relativity.

In special relativity it is enlightening to replace the parameter $\sigma$ in (1) by one that is physically more meaningful. If the $x$-axis moves with constant velocity $V$ relative to the $x'$-axis, for an observer on the $x'$-axis, $x'/t' = V$ is the constant velocity of the origin $x = 0$ of the $x$-axis.

$$\begin{align*}
V \\
| \\
\downarrow \\
x' \\
\downarrow \\
x
\end{align*}$$

But from (1) with $x = 0$

$$V = \frac{x'}{t'} = c \tanh \sigma,$$

so $\sinh \sigma = (V/c)/\sqrt{1 - (V/c)^2}$ and $\cosh \sigma = 1/\sqrt{1 - (V/c)^2}$. We can use this to rewrite the Lorentz transformation (1) in terms of the velocity $V$ as

$$x' = \frac{x + Vt}{\sqrt{1 - (V/c)^2}} \quad t' = \frac{(V/c^2)x + t}{\sqrt{1 - (V/c)^2}}.$$ 

It is physically obvious that to get the inverse transformation just replace $V$ by $-V$.  

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