5. Show that for \( n = 1 \) the solution of (1.8a, b) with \( f(x) = 1 \) for \( x > 0 \), \( f(x) = 0 \) for \( x < 0 \) is given by
\[
 u(x, t) = \frac{1}{2} \left[ 1 + \phi \left( \frac{x}{\sqrt{4t}} \right) \right],
\]
where \( \phi(s) \) is the "error function"
\[
 \phi(s) = \frac{2}{\sqrt{\pi}} \int_0^s e^{-t^2} dt.
\]

6. Show that for \( f(x) \) continuous and of compact support we have \( \lim_{t \to +0} u(x, t) = 0 \) uniformly in \( x \) for the \( u \) given by (1.11).

7. For \( n = 1 \) let \( f(x) \) be bounded, continuous, and positive for all real \( x \).
(a) Show that for the \( u \) given by (1.11)
\[
 |u(\xi + i\eta, t)| < e^{\sqrt{n}/u(\xi, t)}
\]
for real \( \xi, \eta, t \) with \( t > 0 \). [Hint: (1.16).]
(b) Show that
\[
 |u(x, y, t)| < \frac{e^{\sqrt{n/2}}}{\sqrt{2t}} \sup_{|x| < \sqrt{2t}} u(x + y, t)
\]
for \( (x, y, t) \) real, \( t > 0 \). [Hint: Use Cauchy's expression for \( u(x, t) \) as an integral of \( u \) over the circle of radius \( \sqrt{2t} \) and center \( x \) in the complex plane.] This gives a means to estimate the maximum possible age \( t \) of an observed heat distribution \( u \) in terms of its maximum and its gradient, assuming that it has been positive and bounded for a time \( t \).

8. Find all solutions \( u(x, t) \) of the one-dimensional heat equation \( u_t = u_{xx} \) of the form
\[
 u = \frac{1}{\sqrt{t}} f \left( \frac{x}{2\sqrt{t}} \right).
\]
[Hint: \( f(x) \) has to satisfy a linear ordinary second-order equation, of which one solution \( f(x) = e^{-x^2} \) is known, from \( u = K(x, 0, t) \). All others can then be found by quadratures.]

(b) Maximum principle, uniqueness, and regularity
Let \( \omega \) denote an open bounded set of \( \mathbb{R}^n \). For a fixed \( T > 0 \) we form the cylinder \( \Omega \) in \( \mathbb{R}^{n+1} \) with base \( \omega \) and height \( T \):
\[
 \Omega = \{ (x, t) \mid x \in \omega, 0 < t < T \}.
\]
(1.30a)
The boundary \( \partial \Omega \) consists of two disjoint portions, a "lower" boundary \( \partial^- \Omega \) and an "upper" one \( \partial^+ \Omega \) (see Figure 7.1):
\[
 \partial^- \Omega = \{ (x, t) \mid \text{either } x \in \partial \omega, 0 < t < T \text{ or } x \in \omega, t = 0 \},
\]
\[
 \partial^+ \Omega = \{ (x, t) \mid x \in \omega, t = T \}.
\]
(1.30b)
(1.30c)
As in the second-order elliptic case the maximum of a solution of the heat equation in \( \Omega \) is taken on \( \partial \Omega \); but a more subtle distinction between the forward and backwards \( t \)-directions makes itself felt:

\[ \text{Theorem. Let } u \text{ be continuous in } \bar{\Omega} \text{ and } u, u_{x \alpha} \text{ exist and be continuous in } \Omega \text{ and satisfy } u_t - \Delta u < 0. \text{ Then }
\]
\[
 \max_{\bar{\Omega}} u = \max_{\Omega} u.
\]
(1.31)
\[ \text{Proof. Let at first } u_t - \Delta u < 0 \text{ in } \Omega. \text{ Let } \Omega_{\epsilon} \text{ for } 0 < \epsilon < T \text{ denote the set }
\]
\[
 \Omega_{\epsilon} = \{ (x, t) \mid x \in \omega, 0 < t < T - \epsilon \}.
\]
Since \( u \in C^0(\Omega_{\epsilon}) \) there exists a point \( (x, t) \in \bar{\Omega}_{\epsilon} \) with
\[
 u(x, t) = \max_{\bar{\Omega}_{\epsilon}} u.
\]
If here \( (x, t) \in \Omega_{\epsilon} \) the necessary relations \( u_t = 0, \Delta u < 0 \) would contradict \( u_t - \Delta u < 0 \). If \( (x, t) \in \partial^+ \Omega_{\epsilon} \) we would have
\[
 u_t > 0, \quad \Delta u < 0
\]
leading to the same contradiction. Thus \( (x, t) \in \partial^- \Omega_{\epsilon} \) and
\[
 \max_{\bar{\Omega}_{\epsilon}} u = \max_{\partial^- \Omega_{\epsilon}} u = \max_{\partial \Omega_{\epsilon}} u.
\]
(1.31)
Since every point of \( \bar{\Omega} \) with \( t < T \) belongs to some \( \Omega_{\epsilon} \) and \( u \) is continuous in \( \bar{\Omega}, (1.31) \) follows. Let next \( u_{x_t} - \Delta u < 0 \text{ in } \Omega \). Introduce
\[
 v(x, t) = u(x, t) - kt
\]
with a constant positive \( k \). Then \( v_t - \Delta v = u_t - \Delta u - k < 0 \text{ and }
\]
\[
 \max_{\bar{\Omega}} v = \max_{\partial \Omega} (v + kT) = \max_{\partial \Omega} v + kT < \max_{\bar{\Omega}} u + kT.
\]
For \( k \to 0 \) we obtain (1.31). \[ \square \]
The maximum principle immediately yields a uniqueness theorem.

**Theorem.** Let \( u \) be continuous in \( \overline{\Omega} \) and \( u, u_{\infty} \) exist and be continuous in \( \overline{\Omega} \). Then \( u \) is determined uniquely in \( \overline{\Omega} \) by the value of \( u - \Delta u \) in \( \Omega \) and of \( u \) on \( \partial \Omega \).

For the proof it is sufficient to consider the case where \( u - \Delta u = 0 \) in \( \Omega \) and \( u = 0 \) on \( \partial \Omega \). Applying (1.31) to \( u \) and \( -u \) we find that
\[
\max_{\Omega} u = \max_{\Omega} (-u) = 0,
\] (1.32)
and hence that \( u = 0 \) in \( \overline{\Omega} \).

We can extend the maximum principle and the uniqueness theorem to the case where \( \Omega \) is the "slab"
\[
\Omega = \{ (x, t) \mid \alpha \in \mathbb{R}^n, 0 < t < T \},
\] (1.33)
if we assume that \( u \) satisfies a certain growth condition at infinity.

**Theorem.** Let \( u \) be continuous for \( x \in \mathbb{R}^n \), \( 0 < t < T \), and let \( u, u_{x_1} \) exist and be continuous for \( x \in \mathbb{R}^n \), \( 0 < t < T \), and satisfy
\[
u(x, t) < M e^{\alpha |x|} \quad \text{for} \quad 0 < t < T, x \in \mathbb{R}^n
\] (1.34a)
\[
u(x, 0) = f(x) \quad \text{for} \quad x \in \mathbb{R}^n.
\] (1.34b)

Then
\[
u(x, t) < \sup_{x} f(x) \quad \text{for} \quad 0 < t < T, x \in \mathbb{R}^n.
\] (1.35)

It is clear that this theorem implies that the solution of the initial-value problem
\[
u_t - \Delta \nu = 0 \quad \text{for} \quad 0 < t < T
\] (1.36a)
\[
u(x, 0) = f(x)
\] (1.36b)
is unique provided we restrict ourselves to solutions satisfying
\[
|\nu(x, t)| < M e^{\alpha |x|^2} \quad \text{for} \quad 0 < t < T.
\] (1.36c)

This shows that for bounded continuous \( f \) formula (1.11) represents the only bounded solution \( u \) of (1.8a, b). Obviously the Tychonoff solution (1.19), (1.20) for which \( u(x, 0) = 0 \) cannot satisfy an inequality of the type (1.36c). By (1.24) it does satisfy such an inequality with the constant \( a \) replaced by \( 1/\theta t \).

**Proof of the Theorem.** It is sufficient to show (1.35) under the assumption that
\[
4aT < 1
\] (1.37a)

For we can always divide the interval \( 0 \leq t \leq T \) into equal parts, each of length \( \tau < 1/4a \), and conclude successively for \( k=0,1,\ldots, T/\tau \) that
\[
u(x, t) < \sup_{y} u(y, k\tau) < \sup_{y} u(y, 0)
\]
for \( k\tau < t < (k+1)\tau \). Assume then (1.37a). We can find an \( \epsilon > 0 \) such that
\[
4a(T+\epsilon) < 1.
\] (1.37b)

Given a fixed \( y \) we consider for constants \( \mu > 0 \) the functions
\[
\nu_{\mu}(x, t) = u(x, t) - \mu (4\pi(T+\epsilon-t))^{-n/2} \exp\left(\frac{|x-y|^2}{4(T+\epsilon-t)}\right)
\]
(1.38)
defined for \( 0 < t < T \). Since \( K(x, y, t) \) as defined by (1.10o), with \( |x-y|^2 \) replaced by \( |x-y| \cdot |x-y| \), satisfies \( K_t = \Delta K \) for any complex \( x, y, t \) with \( t \neq 0 \), we find that
\[
\frac{\partial}{\partial t} \nu_{\mu} = \nu_t - \Delta \nu_{\mu} = u_t - \Delta u \leq 0.
\] (1.39)

Consider the "circular" cylinder
\[
\Omega = \{ (x, t) \mid |x-y| < \rho, 0 < t < T \}
\] (1.40)
of radius \( \rho \). Then by (1.31)
\[
\nu_{\mu}(y, t) < \max_{\partial \Omega} \nu_{\mu} .
\] (1.41)

Here on the plane part of \( \partial \Omega \), since \( \mu K > 0 \),
\[
\nu_{\mu}(x, 0) < u(x, 0) < \sup_{x} f(x).
\] (1.42a)

On the curved part \( |x-y| = \rho, 0 < t < T \) of \( \partial \Omega \) by (1.38), (1.34b), (1.37b)
\[
\nu_{\mu}(x, t) = \frac{4\pi}{\theta t} - \mu (4\pi(T+\epsilon-t))^{-n/2} \exp\left(\frac{|x-y|^2}{4(T+\epsilon-t)}\right)
\]
\[
< M e^{\alpha |x|^2} \frac{4\pi}{\theta t} - \mu (4\pi(T+\epsilon-t))^{-n/2} e^{\alpha |x|^2/4(T+\epsilon-t)}
\]
\[
< \sup_{x} f(x)
\]
for all sufficiently large \( \rho \). Thus
\[
\max_{\partial \Omega} \nu_{\mu} < \sup_{x} f(x).
\]

It follows from (1.41), (1.38) that
\[
\nu_{\mu}(y, t) = u(x, t) - \mu (4\pi(T+\epsilon-t))^{-n/2} < \sup f(x)
\]
For \( \mu \to 0 \) we obtain (1.35).

\( \square \)

In order to derive regularity properties of a solution of the heat equation in a bounded region we make use of Green's identity, as was done for harmonic functions on p. 76. Let again \( \Omega \) denote the cylindrical region (1.30a), where \( \omega \) is a bounded open set in \( \mathbb{R}^n \) with sufficiently regular
boundary. Let \( u, u_{\gamma, x}, \) exist and be continuous in \( \bar{\Omega} \) and satisfy \( u - \Delta u = 0 \). For an arbitrary function \( v(x,t) \in C^2(\bar{\Omega}) \) we find by integration by parts that

\[
0 = \int_\Omega v(u, \Delta u) \, dx = - \int_\Omega u_v (v_\Delta t) \, dx + \int_{t=0}^T \int_{\Omega \times \omega} v u \, dx dt - \int_{t=0}^T \int_{\Omega \times \omega} v u \, dx dt
\]

(1.43)

For a certain \( \xi \in \omega \) and \( \epsilon > 0 \) we choose

\[
v(x,t) = K(x, \xi, T + \epsilon - t),
\]

so that \( v_\xi + \Delta v = 0 \). Then for \( \epsilon \to 0 \)

\[
\int_{x \in \omega} v u \, dx = \int_{x \in \omega} K(x, \xi, \epsilon) u(x, T) \, dx \to u(\xi, T),
\]

(1.44a)

since by the theorem of p. 169

\[
w(\xi, \epsilon) = \int K(\xi, x, \epsilon) u(x, T) \, dx = \int K(\xi, x, \epsilon) u(x, T) \, dx
\]

is a solution of \( w_t - \Delta w = 0 \) with initial values

\[
w(\xi, 0) = u(x, T).
\]

Since also \( K(x, \xi, T + \epsilon - t) \) is uniformly continuous in \( \epsilon, x, t \) for \( \epsilon > 0 \), \( x \in \partial \omega \), \( 0 < t < T \) and for \( x \in \omega \), \( t = 0 \), we find from (1.43) that

\[
u(x, T) = \int_\omega K(x, \xi, T) u(x, 0) \, dx
\]

\[
+ \int_{t=0}^T \int_{x \in \partial \omega} \left( K(x, \xi, T - t) - u(x, t) \frac{dK(x, \xi, T - t)}{dn} \right) \, ds
\]

(1.45)

We use this formula to extend \( u(x, \xi) \) to complex \( \xi \)-arguments \( \xi = \eta + i \xi \) (with \( \eta, \xi \) real), keeping \( T \) real. The first integral in (1.45) trivially is an entire analytic function of \( \xi \). Moreover for \( 0 < t < T \), \( x \neq \eta \)

\[
K(x, \xi, T - t) = (4\pi(T - t))^{-\eta/2} \exp \left[ - (x - \xi) \cdot (x - \xi) / 4(T - t) \right]
\]

is analytic in \( \xi \) and (see (1.16)) bounded in absolute value by

\[
(4\pi(T - t))^{-\eta/2} \exp \left[ \frac{\| \eta \|^2 - |x - \eta|^2}{4(T - t)} \right].
\]

*The fact that \( x \) is integrated only over the region \( \omega \) instead of over all of \( \mathbb{R}^n \) does not change the proof given on p. 169, as long as \( \xi \) is a fixed point of \( \omega \).

Thus \( K(x, \xi, T - t) \) is bounded uniformly for complex \( \xi = \eta + i \xi \) as long as \( |x - \eta|^2 - |\xi|^2 \) is bounded below by a positive constant. The same holds for \( dK/\partial n \). In the second integral we first extend the \( \xi \)-integration from \( 0 \) to \( T - \epsilon \) and then let \( \epsilon \to 0 \). Since sequences of analytic functions which converge uniformly in a complex region have analytic limits, it follows that \( u(\xi, t) \) is analytic in \( \xi \), as long as \( |x - \eta|^2 - |\xi|^2 > 0 \) for all \( x \in \partial \omega \). This is certainly the case for complex \( \xi \) near a real point of \( \omega \).

It follows that \( u(\xi, T) \) is real analytic for \( \xi \in \omega \). More precisely \( u(\xi, T) \) is analytic for those complex \( \xi \) for which \( \text{Re}\xi \) is less than the distance of \( \text{Re}\xi \) from \( \partial \omega \). Hence a solution \( u(x, t) \) of \( u_t - \Delta u = 0 \) is real analytic in \( x \) in any open set where \( u \) and the \( u_{\gamma, x} \) are continuous. Moreover \( u(x, t) \) will be an entire function of \( x \) if defined for all real \( x \) and for \( t \) restricted to an open interval.

We easily conclude that solutions of \( u_t - \Delta u = 0 \) in an open set \( \Omega \in \mathbb{R}^{n+1} \) belong to \( C^\omega(\Omega) \). For if \( u \) has continuous \( x \)-derivatives of all orders, then \( u_{\gamma, x} = \Delta u_{\gamma, x} \) is continuous, and hence equals \( u_{\gamma, x} \). Thus \( v = u_{\gamma, x} \) is also a solution of the heat equation with the same regularity properties as \( u \). The same holds again for \( v_{\gamma, x} = u_{\gamma, x} \) and then also for \( \Delta u = u_{\gamma, x} \). Thus \( u_t = \Delta u \) is continuous. Proceeding in this manner yields that all derivatives of \( u(x, t) \) are continuous. As observed earlier analyticity of \( u(x, t) \) with respect to \( t \) cannot be expected. This fits in with the idea that the future of a heat distribution does not depend exclusively on the past, but also on outside influences that cannot be predicted.

**Problem**

Let \( u \) be a solution of the one-dimensional heat equation \( u_t = u_{xx} \) in an open subset \( \Omega \) of the xt-plane. Show that at a point of \( \Omega \) there exist constants \( A, M \) such that

\[
\left| \frac{\partial u}{\partial x} \right| < AM^x(2\lambda)!
\]

(1.46)

for all nonnegative integers \( k \). [Hint: Use that \( u \) is analytic in \( x \).]

(c) A mixed problem

For \( n = 1 \) let \( u(x, t) \) be a solution of \( u_t - u_{xx} = 0 \) in a half strip

\[
0 < x < L \quad 0 < t.
\]

(1.47)

We seek the \( u \) satisfying the boundary conditions

\[
u(0, t) = u(L, t) = 0 \quad \text{for } t > 0
\]

(1.48a)

and initial condition

\[
\text{initial}
\]

\[
(0, t) = x = f(x) \quad 0 < x < L.
\]

(1.48b)

Here \( u \) might represent the temperature in an insulated rod with the ends held at a constant temperature. This problem could be solved by Fourier