1) p.67 #10

$u_{tt} = 9u_{xx}$ in $0<x<\pi/2$, $u(x,0)=\cos(x) = \varphi$

$u_t(x,0) = 0 = \psi$, $u_x(0,t)=0$

$u(\pi/2, t) = 0$

a) Want even w.r.t. 0

odd w.r.t. $\pi/2$

$\varphi_{ext}(x) = \varphi_{ext}(-x)$ $\qquad$ $\varphi_{ext}(2l-x) = -\varphi_{ext}(x)$

$\underbrace{\cos(x) = \cos(-x) \checkmark \qquad \cos(\pi - x) = -\cos(x) \checkmark}_{\varphi \text{ satisfies these already!}}$

so $\varphi_{ext} = \varphi = \cos(x)$.

so on whole line

$$v(x,t) = \frac{1}{2}[\cos(x+ct) + \cos(x-ct)] - \frac{1}{2c}\int 0$$

solves this...

Restrict to just $0<x<\pi/2$ doesn't change anything since $\varphi_{ext} = \varphi$.

$$\Rightarrow u(x,t) = \frac{1}{2}[\cos(x+3t) + \cos(x-3t)]$$

b) $u(x,t) = X(x)T(t)$

w/

$X'' + \beta^2 X = 0$ $\qquad$ $T'' + 3\beta^2 T = 0$

$X(x) = C\cos(\beta x) + D\sin(\beta x)$

$T(t) = A\cos(3\beta t) + B\sin(3\beta t)$

$X(\frac{\pi}{2}) = C\cos(\beta \frac{\pi}{2}) + D\sin(\beta \pi/2)$

$$X'(x) = -C\beta \sin(\beta x) + D\beta \cos(\beta x)$$

$$X'(0) = D\beta \overset{1}{\cos(\beta \cdot 0)} = 0$$

$$\Rightarrow D = 0$$

$$\Rightarrow C\cos\left(\beta \frac{\pi}{2}\right) = 0$$

$$\Rightarrow \beta \frac{\pi}{2} = \left(n + \frac{1}{2}\right)\pi$$

$$\beta = 2n+1$$

$$T'(0) = 0 \Rightarrow -3\beta A \overset{0}{\sin(3\beta \cdot 0)} + 3\beta B \overset{1}{\cos(3\beta \cdot 0)} = 0$$

$$B = 0.$$

$$\Rightarrow u(x,t) = \sum_n A_n \cos((6n+3)t) \cos((2n+1)x)$$

$$u(x,0) = \sum_n A_n \cos((2n+1)x) = \cos(x)$$

$$\Rightarrow A_n = 0 \ \forall n \neq 0$$

$$A_0 = 1$$

$$\Rightarrow u(x,t) = \cos(3t)\cos(x)$$

$\square$

2) p.70 #1

$u_t - ku_{xx} = f(x,t)$ $\quad\quad 0<x<\infty$

$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad 0<t<\infty$

$u(0,t)=0$

$u(x,0)=\varphi(x)$

$$\varphi_{ext}(x) = \begin{cases} \varphi(x) & x \ge 0 \\ -\varphi(-x) & x<0 \end{cases}$$ Odd extension

$$f_{ext}(x) = \begin{cases} f(x,t) & x \ge 0 \\ -f(-x,t) & x<0 \end{cases}$$

On $\mathbb{R}$,

$$u(x,t) = \underbrace{\int_{-\infty}^{\infty} S(x-y,t)\varphi_{ext}(y)\,dy}_{1} + \underbrace{\int_0^t \int_{-\infty}^{\infty} S(x-y,t-s) f_{ext}(y,s)\,dy\,ds}_{2}$$

$$1 = \int_0^\infty S(x-y,t)\varphi(y)dy + \int_{-\infty}^0 -S(x-y,t)\varphi(-y)\,dy$$

$y \to -y$

$$= \int_0^\infty S(x-y,t)\varphi(y)dy + \int_\infty^0 S(x+y,t)\varphi(y)dy$$

$$= \int_0^\infty [S(x-y,t) - S(x+y,t)]\varphi(y)\,dy$$

Similar for 2

$$\Rightarrow u(x,t) = \int_0^\infty [S(x-y,t) - S(x+y,t)]\varphi(y)dy +$$

$$\int_0^t \int_0^\infty [S(x-y,t-s) - S(x+y,t-s)]\, f(y,s)\,dy\,ds$$

3) p. 71 #2

$$v_t - kv_{xx} = f(x,t)$$

$$v(0,t) = h(t) \qquad 0<x<\infty, \ 0<t<\infty$$

$$v(x,0) = \varphi(x)$$

Let $V(x,t) = v(x,t) - h(t)$

$$V_t - kV_{xx} = f(x,t) - h'(t)$$

$$V(0,t) = 0$$

$$V(x,0) = \varphi(x) - h(0)$$

$\Rightarrow$ by (2)

$$V(x,t) = \int_0^\infty \left[S(x-y,t) - S(x+y,t)\right]\left(\varphi(y) - h(0)\right)dy$$
$$+ \int_0^t\int_0^\infty \left[S(x-y,t-s) - S(x+y,t-s)\right]\left(f(y,s) - h'(s)\right)dy\,ds$$

$\Rightarrow v(x,t) = V(x,t) + h(t)$

4) $u''(t) + a^2 u(t) = f(t)$

$(u(0) = \varphi,\ u'(0) = \psi)$

IF $f \equiv \varphi \equiv 0$

$u(t) = \frac{1}{a}\sin(ta)\psi$ is the solution.

So by Duhamel,

$$u(t) = S'(t)\varphi + S(t)\psi + \int_0^t S(t-s) f(s)\, ds$$

where $S(t) = \frac{1}{a}\sin(ta)$

$\Rightarrow S'(t) = \cos(ta)$

$$\Rightarrow \boxed{u(t) = \cos(ta)\varphi + \frac{1}{a}\sin(ta)\psi + \int_0^t \frac{1}{a}\sin((t-s)a)\, f(s)\, ds}$$

$\square$

5) p.79 #2.

$$u_{tt} = c^2 u_{xx} + e^{\alpha x} \qquad u(x,0) = 0 \qquad u_t(x,0) = 0$$

By (3)

$$u(x,t) = \frac{1}{2c} \iint_{\Delta} e^{\alpha x} = \frac{1}{2c} \int_0^t \int_{x-c(t-s)}^{x+c(t-s)} e^{\alpha y}\, dy\, ds$$

$$= \frac{1}{2c} \int_0^t \frac{1}{\alpha} e^{\alpha y} \Big|_{x-c(t-s)}^{x+c(t-s)} ds$$

$$= \frac{1}{\alpha 2c} \int_0^t e^{\alpha(x+c(t-s))} - e^{a(x-c(t-s))}\, ds$$

$$= \frac{e^{\alpha x}}{\alpha 2c} \left[\int_0^t e^{c(t-s)} - e^{c(s-t)}\, ds \right]$$

$$= \frac{e^{\alpha x}}{\alpha 2c} \left[\frac{1}{-c} e^{c(t-s)} - \frac{1}{c} e^{c(s-t)} \Big|_0^t \right]$$

$$= \frac{e^{\alpha x}}{\alpha 2c} \left[-\frac{1}{c} + e^{ct} - \frac{1}{c} + e^{-ct} \right]$$

$$= \frac{e^{\alpha x}}{\alpha 2c} \left[-\frac{2}{c} + e^{ct} + e^{-ct} \right]$$

$\square$

7) p. 80 # 11

$$u(x,t) = \begin{cases} h(t - \frac{x}{c}) & x < ct \\ 0 & x \geq ct \end{cases}$$

solves on $(0, \infty)$

with $u(x,0) = 0$ and $u(0,t) = h(t)$

$u_{tt} = h''(t - \frac{x}{c})$

$u_{xx} = h''(t - \frac{x}{c})\left(\frac{1}{c^2}\right)$

$\therefore u_{tt} = c^2 u_{xx}$ for $x < ct$.

For $x \geq ct$

$u(x,t) = 0$ so trivial.

$u(0,t) = h(t)$

$\parallel$

$h(t - \frac{0}{c}) = h(t)$ ✓

$u(x,0) = 0$ as $x \geq 0$.

$\square$

8) p. 89 #1

a) So frequency in (9) is $\frac{n\pi x}{\ell}$. Replacing ℓ with $\ell/2$

we obtain frequency $\frac{2n\pi x}{\ell}$ which is twice frequency of one octave.

b) String tightened so higher frequency as in part a).

9) p. 89 #4

$u_{tt} = c^2 u_{xx} - r u_t$ $\quad 0<x<\ell$

$u = 0$ $\quad$ at ends

$u(x,0) = \phi(x)$ $\quad 0<r<\frac{2\pi c}{\ell}$

$u_t(x,0) = \psi(x)$

$u(x,t) = X(x)T(t)$

$X(t)T''(t) = c^2 X''(x)T(t) - r X(x) T'(t)$

Divide by $c^2 XT$

$$\frac{1}{c^2}\frac{T''(t)}{T} = \frac{X''(x)}{X} - \frac{r}{c^2}\frac{T'(t)}{T}$$

$$\left(\frac{1}{c^2}\frac{T''}{T} + \frac{r}{c^2}\frac{T'}{T}\right) = \frac{X''}{X} = -\lambda$$

$$\frac{X''}{X} = -\beta^2$$

$$X'' + \beta^2 X = 0$$

$$X(x) = C\cos(\beta x) + D\sin(\beta x)$$

$$X(0)=0 \text{ so } C=0$$

$$X(\ell)=0 \text{ so } \beta = \frac{n\pi}{\ell}$$

$$\Rightarrow X_n(x) = \sin\left(\frac{n\pi x}{\ell}\right)$$

For T,

$$T'' + rT' + \beta^2 c^2 T = 0$$

roots $m = \dfrac{-r \pm \sqrt{r^2 - 4\beta^2 c^2}}{2} = \dfrac{-r \pm \sqrt{r^2 - \frac{4n^2\pi^2c^2}{\ell^2}}}{2}$ $\quad \alpha_n$

as $0 < r < \dfrac{2\pi c}{\ell}$

so for $n > 0$ this is an imaginary number

$\Rightarrow$ for $n>0$

$$T_n(t) = e^{-\frac{r}{2}t}\left(A_n\cos(\alpha_n t) + B_n\sin(\alpha_n t)\right)$$

$$\Rightarrow u(x,t) = e^{-r/2\,t}\sum_n \left(A_n\cos(\alpha_n t) + B_n\sin(\alpha_n t)\right)\sin\left(\frac{n\pi x}{\ell}\right)$$

$$\varphi(x) = u(x,0) = \cancel{e}\sum_n A_n\sin\left(\frac{n\pi x}{\ell}\right)$$

$$\Rightarrow A_n = \frac{2}{\ell}\int_0^\ell \varphi(x)\sin\left(\frac{n\pi x}{\ell}\right)dx$$

$$\psi(x) = u_t(x,0) = \sum_n B_n\alpha_n \sin\left(\frac{n\pi x}{\ell}\right) dx$$

$$\Rightarrow B_n = \frac{2}{\ell\alpha_n}\int_0^\ell \psi(x)\sin\left(\frac{n\pi x}{\ell}\right)dx$$

10) p. 92 #2

$u_{tt} = c^2 u_{xx}$ $\quad 0 < x < l$

$u_x(0,t) = 0$

$u(l,t) = 0$

a) $X'(0) = 0$

$X(l) = 0$ $\quad X'' + \lambda X = 0$

$X(x) = C\cos\beta x + D\sin\beta x$

$X'(x) = -C\beta\sin\beta x + D\beta\cos\beta x$

$X'(0) = D\beta = 0 \Rightarrow D = 0$

$X(l) = C\cos(\beta l) = 0$

$\Rightarrow \beta l = \pi n + \frac{\pi}{2}$

$\Rightarrow \beta = \frac{(n+\frac{1}{2})\pi}{l}$

$\Rightarrow$ eigenfunction is $\cos\left(\frac{(n+\frac{1}{2})\pi x}{l}\right)$

b)

$$u(x,t) = \sum_n \left(A_n \cos\left(\frac{n\pi c t}{l}\right) + B_n \sin\left(\frac{n\pi c t}{l}\right)\right) \cos\left(\frac{(n+\frac{1}{2})\pi x}{l}\right)$$