Problem Set 8

DUE: Thurs. Mar. 26 in class. [Late papers will be accepted until 1:00 PM Friday.]

This week. Please read the Chapters 7 and 9 in the Strauss text.

[Lots of problems. Again, fortunately, most of them are short.]

1. Strauss, p. 172 #1

2. Strauss, p. 172 #2

3. Strauss, p. 175 #1

4. Solve $\Delta u = 0$ in the annulus $1 \leq x^2 + y^2 \leq 2$ with $u(x,y) = 1$ on the circle $x^2 + y^2 = 1$ and $u(x,y) = 7$ on $x^2 + y^2 = 2$.

5. Suppose $u$ is a twice differentiable function on $\mathbb{R}$ which satisfies the ordinary differential equation

$$u'' + b(x)u' - c(x)u = 0,$$

where $b(x)$ and $c(x)$ are continuous functions on $\mathbb{R}$ with $c(x) > 0$ for every $x \in (0,1)$.

a) Show that $u$ cannot have a positive local maximum in the interval $(0,1)$, that is, have a local maximum at a point $p$ where $u(p) > 0$. Also show that $u$ cannot have a negative local minimum in $(0,1)$.

[The example $u'' + u = 0$ has $u(x) = \sin x$ as a solution, which does have positive local maxima and negative local minima. This shows that some assumption, such as our $c(x) > 0$ is needed.]

b) If $u(0) = u(1) = 0$, prove that $u(x) = 0$ for every $x \in [0,1]$.

c) If $u$ satisfies

$$4u_{xx} + 3u_{yy} - 5u = 0$$

in a region $\mathcal{D} \subset \mathbb{R}^2$, show that it cannot have a local positive maximum. Also show that $u$ cannot have a local negative minimum.

d) Repeat the above for a solution of

$$4u_{xx} - 2u_{xy} + 3u_{yy} + 7u_x + u_y - 5u = 0.$$ 

[Remark: If $A = (a_{ij})$ and $B = (b_{ij})$ are positive semi-definite symmetric $n \times n$ matrices, then $\sum_{i,j=1}^{n} a_{ij}b_{ij} \geq 0$.]

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e) If a function \( u(x, y) \) satisfies the above equation in a bounded region \( D \subseteq \mathbb{R}^2 \) and is zero on the boundary of the region, show that \( u(x, y) \) is zero throughout the region.

6. Consider the Dirichlet problem \( \Delta u - 5u = 0 \) in a bounded region \( \Omega \subseteq \mathbb{R}^2 \) with \( u(x, y) = f(x, y) \) for points \( (x, y) \) on the boundary \( \partial \Omega \). Prove the uniqueness in two ways: using a maximum principle (see the previous problem) and using an energy argument.

7. a) Let \( B \) be the ball \( \{r^2 = x^2 + y^2 + z^2 < a^2\} \) in \( \mathbb{R}^3 \). Compute all the radial eigenfunctions \( u(r) \) of \( -\Delta \) with Neumann boundary conditions \( \partial u / \partial r = 0 \) for \( r = a \). Thus, you are solving \( -[u_{rr} + \frac{2}{r}u_r] = \lambda u \). [SUGGESTION: the substitution \( v(r) = ru(r) \) is useful. Note it implies \( v(0) = 0 \).]

b) Compute the corresponding eigenvalues (there is an explicit formula).

c) Use this to solve the heat equation \( u_t = \Delta u \) in \( B \) with \( u_r = 0 \) on the boundary in the special case where the initial temperature, \( u(x, 0) = \varphi(r) \) depends only on \( r \). Your solution will be an infinite series. Please include a formula for finding the coefficients.

8. Strauss, p. 184 #2

9. Strauss, p. 184 #5

10. Strauss, p. 187 #1

11. Strauss, p. 187 #2

12. Strauss, p. 190 #2

[Last revised: March 21, 2015]