1) p. 172 #1

$\Delta u = 0$ in $D = \{r < 2\}$

$u = 3\sin(2\theta) + 1$ for $r = 2$

a) Find max u in $\overline{D}$

Max occurs on bd.

$\therefore$ max $u = 3(1) + 1 = 4$ (i.e. $\theta = \pi/4$)

b) Find $u(0)$

$$u(0) = \frac{1}{2\pi}\int_0^{2\pi} 3\sin(2\theta) + 1 \; d\theta$$

$u = 2\theta$
$du = 2d\theta$

$$= \frac{1}{4\pi}\int_0^{4\pi} 3\sin(u) + 1 \; du$$

$$= \frac{1}{4\pi}\left[-3\cos(u) + u \Big|_0^{4\pi}\right]$$

$$= \frac{1}{4\pi}\left[\cancel{-3} + 4\pi - (\cancel{-3} + 0)\right] = \underline{1}$$

2) p. 172 #2

$\Delta u = 0 \quad \{r<a\}$

$u = 1 + 3\sin\theta$ when $r=a$

By Poisson's formula:

$$u(r,\theta) = (a^2 - r^2)\int_0^{2\pi} \frac{1 + 3\sin\varphi}{a^2 - 2ar\cos(\theta-\varphi) + r^2}\,\frac{d\varphi}{2\pi}$$

Not helpful.

So let's use series,

$$A_n = \frac{1}{\pi a^n}\int_0^{2\pi}(1+3\sin\varphi)\cos(n\varphi)\,d\varphi$$

$$= \frac{1}{\pi a^n}\int_0^{2\pi}\cos(n\varphi)\,d\varphi + \frac{3}{\pi a^n}\underbrace{\int_0^{2\pi}\sin\varphi\cos(n\varphi)\,d\varphi}_{\text{nonzero only for } n=0 \text{ by orthogonality}}$$

$$A_n = 0 + 0 = 0$$

except:

$$A_0 = \frac{1}{\pi}\int_0^{2\pi} 1 + 3\sin\varphi\,d\varphi = \frac{1}{\pi}\left[\varphi - 3\cos\varphi\right]\Big|_0^{2\pi}$$

$$= \frac{1}{\pi}\left[2\pi - 3 - (0-3)\right] = 2 \qquad \left(\frac{A_0}{2} = 1\right)$$

$$B_n = \frac{1}{\pi a^n}\int_0^{2\pi}(1+3\sin\varphi)\sin(n\varphi)\,d\varphi$$ only nonzero for $n=1$ by orthogonality

$$B_1 = \frac{1}{\pi a}\int_0^{2\pi}\sin\varphi + 3\sin^2\varphi\,d\varphi = \frac{3}{a}$$

$$\therefore\ u = 1 + \left(\frac{3r}{a}\right)\sin\theta$$

$\square$

3) p. 175 #1

$u = 1 + 3\sin\theta$ on $r = a$

u bounded as $r \to \infty$

$$u(r,\theta) = \frac{1}{2}(C_0 + D_0 \log r) + \sum_{n=1}^{\infty} \Big((C_n r^n + D_n r^{-n}) \cos n\theta$$

$$+ (A_n r^n + B_n r^{-n}) \sin(n\theta) \Big)$$

as bounded as $r \to \infty \Rightarrow D_0, C_n, A_n = 0$

$$u(r,\theta) = \frac{1}{2} C_0 + \sum_{n=1}^{\infty} D_n r^{-n} \cos(n\theta) + B_n r^{-n} \sin(n\theta)$$

Exact same as p. 172 #2 now

i.e. $B_1 = \frac{3}{a}$ $\quad C_0 = 2 \quad D_n, B_n = 0$

$$\Rightarrow u(r,\theta) = 1 + \frac{3a}{r} \sin\theta$$

4) Δu in $1 \le x^2+y^2 \le 2$

w/ $u(x,y)=1$ for $r=1$

$u(x,y)=7$ for $r=2$

$$u(1,\theta)=\frac{1}{2}C_0+\sum_{n=1}^{\infty}(C_n+D_n)\cos(n\theta)+(A_n+B_n)\sin(n\theta)=1$$

$$\Rightarrow C_0=2$$

$$\left(\begin{matrix} C_n+D_n=0 \\ A_n+B_n=0 \end{matrix}\right) (\star)$$

$$7=u(2,\theta)=1+D_0\log 2+\sum_{n=1}^{\infty}(C_n 2^n+D_n 2^{-n})\cos(n\theta)+(A_n 2^n+B_n 2^{-n})\sin(n\theta)$$

Again,

$$\left(\begin{matrix} C_n 2^n+D_n 2^{-n}=0 \\ A_n 2^n+B_n 2^{-n}=0 \end{matrix}\right) (\star\star)$$

$$7=1+D_0\log 2$$

$$D_0=\frac{6}{\log 2}$$

$$\star+\star\star \Rightarrow A_n, B_n, C_n, D_n=0.$$

$$\Rightarrow \boxed{u(r,\theta)=1+\frac{6}{\log 2}\log r}$$

5) $u'' + b(x)u' - c(x)u = 0$

$c(x) > 0 \quad \forall x \in (0,1)$

a) Show u cannot have pos. local maximum in $(0,1)$.

If so at $x_0 \in (0,1)$

$u''(x_0) < 0$ and $u'(x_0) = 0$ and $u(x_0) > 0$

$\Rightarrow \underbrace{u''(x_0)}_{<0} - c(x_0)\underbrace{u(x_0)}_{>0} = 0$ so impossible! Same for neg min.

b) $u(0) = u(1) = 0$

$\therefore$ as no where can be pos. max or neg. min

and u is cont. $\Rightarrow u = 0$ in $(0,1)$

c) $4u_{xx} + 3u_{yy} - 5u = 0$

local max $\Rightarrow u_{xx}u_{yy} - u_{xy}^2 > 0$ pos $\Rightarrow u > 0$

$u_{xx} < 0$

$\therefore \underbrace{4u_{xx}}_{<0} + \underbrace{3u_{yy}}_{<0} - \underbrace{5u}_{>0} = 0$ impossible!

as

$u_{xx}u_{yy} - u_{xy}^2 > 0$

Same for neg. min.

d) $4u_{xx} - 2u_{xy} + 3u_{yy} + 7u_x + u_y - 5u = 0$

Local $\boxed{\text{neg min}}$

$u_{xx}u_{yy} - u_{xy}^2 > 0$

$u_{xx} > 0 \Rightarrow u_{yy} > 0$

$u_x = u_y = 0$

$u < 0$

$$4u_{xx} - 2u_{xy} + 3u_{yy} - 5u = 0$$

$$A = \begin{pmatrix} u_{xx} & u_{xy} \\ u_{xy} & u_{yy} \end{pmatrix} \qquad B = \begin{pmatrix} 4 & -1 \\ -1 & 3 \end{pmatrix}$$

A is pos semi definite
as local max.

B is pos def.

so by hint: $\underbrace{4u_{xx} - 2u_{xy} + 3u_{yy}}_{\geq 0} \underset{<0}{-5u} = 0$ Impossible.

e) Same as b)

6) $\Delta u - 5u = 0$ in Ω

$u(x,y) = f(x,y)$ on $\partial\Omega$

a) No local pos. max or thin $w = u - v$ where u, v two solutions.

$$-5u < 0$$

$$\Delta u = u_{xx} + u_{yy} < 0$$

No local neg min so the

$$-5u > 0$$

$$\Delta u = u_{xx} + u_{yy} > 0$$

$\therefore$ by 5e) $u \equiv 0$.

b) $w = u - v$

$$\Delta w - 5w = 0 \Rightarrow \frac{\Delta w}{5} = w \quad \text{and}$$

$$5\iint w^2\,dx = \iint_\Omega w \Delta w\,dx = \iint_{\partial\Omega} w \frac{\partial w}{\partial n} - \iint_\Omega |\nabla w|^2 dx$$

$$\Rightarrow 5\iint \underbrace{w^2}_{\geq 0} dx = -\iint_\Omega \underbrace{|\nabla w|^2}_{\leq 0} dx$$

so $= 0$!

$\square$

7) $-\Delta$ operator with $\frac{\partial u}{\partial r} = 0$ for $r=a$.

a) Solving $-[u_{rr} + \frac{2}{r} u_r] = \lambda u$

let $v(r) = r u(r)$

$$u = \frac{v}{r}$$

$$u_r = \frac{v_r}{r} - \frac{v}{r^2}$$

$$u_{rr} = \frac{v_{rr}}{r} - \frac{2v_r}{r^2} + \frac{2v}{r^3}$$

$$-\left[\frac{v_{rr}}{r} \cancel{- \frac{2v_r}{r^2}} \cancel{+ \frac{2v}{r^3}} \cancel{+ \frac{2v_r}{r^2}} \cancel{- \frac{2v}{r^3}}\right] = \frac{\lambda v}{r}$$

$$-\frac{v_{rr}}{r} = \frac{\lambda v}{r}$$

$$-v_{rr} = \lambda v$$

with bd conditions, $v(0) = 0$ (i)

$v_r(a) - \frac{v(a)}{a} = 0$ (ii)

$\lambda > 0$. let $\beta = \sqrt{\lambda}$,

$\Rightarrow$ $v = C_1 \cos(\beta r) + C_2 \sin(\beta r)$

Bd cond. (i) $\Rightarrow$ $C_1 = 0$.

Bd cond (ii) $\Rightarrow$ $C_2 \beta \cos(\beta a) - \frac{C_2}{a} \sin(\beta a) = 0$

$\Rightarrow \tan(\beta a) = \beta a$

β_n are positive solutions.

$\lambda < 0$ let $\beta = \sqrt{|\lambda|}$

$$V = C_1 e^{\beta r} + C_2 e^{-\beta r}$$

$$0 = C_1 + C_2$$

$$0 = (\beta - \tfrac{1}{a}) C_1 e^{\beta a} - (\beta + \tfrac{1}{a}) C_2 e^{-\beta a} = 0$$

$$\Rightarrow e^{2\beta a} = \frac{1 + a\beta}{1 - a\beta}$$

As LHS > 0, $0 \leq a\beta < 1$

$\Rightarrow$ This eqn. only has sol'n $\beta = 0$ which is degenerate! So $\lambda > 0$ only sol'ns!

b). $\lambda_n = \beta_n^2$ which solved $\tan(a\beta_n) = a\beta_n$.

c) $$u = \sum_{n=1}^{\infty} \underbrace{\frac{1}{r} A_n \sin(\beta_n r)}_{V \text{ from part a)}} \underbrace{e^{-\lambda_n t}}_{\text{heat part}}$$

wh...

8) $$\iint_{\partial D} u\frac{\partial u}{\partial n}\,dS = \iiint |\nabla u|^2 dx + \iiint u\Delta u\,dx$$

$\Delta u = 0$ on D and $\frac{\partial u}{\partial n} = 0$ on ∂D

$$\Rightarrow 0 = \iiint |\nabla u|^2 dx \Rightarrow u \equiv \text{const}$$

where $u = v - w$ two solutions.

9) $w = u - v$.

$$E(w) = \frac{1}{2}\iint_D |\nabla(u-v)|^2 dx - \iint_{\partial D} h(u-v)\,dS$$

$$= E[u] + \frac{1}{2}\iiint |\nabla v|^2 + \underbrace{\iint_{\partial D} hv\,dS}_{\text{Green}} - \iiint_D \nabla u\cdot\nabla v$$

$$= E[u] + \frac{1}{2}\iiint |\nabla v|^2 + \cancel{\iiint \nabla u\cdot\nabla v} + u\cancel{\Delta u}^{\,0} - \cancel{\iiint \nabla u\cdot\nabla v}$$

$$= E[u] + \frac{1}{2}\iiint |\nabla v|^2$$

Minimized when $\nabla v = 0 \Rightarrow v \equiv \text{const}$.

10) Divide D into $B(x_0,\varepsilon)$ and $D\setminus B(x_0,\varepsilon) = D_\varepsilon$. Set $x_0 = 0$ for simplicity (use coordinate system)

Let $v(x) = \frac{1}{2\pi}\log|x - x_0| = \frac{\log r}{2\pi}$

$$\int_{\partial D_\varepsilon}\left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\right)dS = \int_{\partial D}\left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\right)dS - \int_{B(x_0,\varepsilon)}\left(u\frac{\partial v}{\partial r} - v\frac{\partial u}{\partial r}\right)$$

$$\frac{\partial v}{\partial r} = \frac{\partial}{\partial r}\frac{\log r}{2\pi} = \frac{1}{2\pi r}$$

$$\text{RHS} = \frac{1}{2\pi\varepsilon}\int_{B(x_0,\varepsilon)} u\,dS - \frac{\log\varepsilon}{2\pi}\int_{B(x_0,\varepsilon)}\frac{\partial u}{\partial r} = \bar{u} - \varepsilon\log\varepsilon\,\frac{\partial\bar{u}}{\partial r}$$

as $\varepsilon \to 0$ $\quad \varepsilon \log \varepsilon \to 0$ by L'Hopital's rule

$$\Rightarrow u(x_0) = \frac{1}{2\pi} \int_{\partial D} \left[u \frac{\partial}{\partial n} \log|x - x_0| - \frac{\partial u}{\partial n} \log|x - x_0| \right] ds$$

11) Use Green's 2^{ND}

$u = \varphi \quad v = \frac{1}{|x|}$ on $D \setminus B(0, \varepsilon)$

$$\iiint_D \left(\varphi \Delta \frac{1}{|x|} - \frac{1}{|x|} \Delta \varphi \right) dx = \iint_{\partial D} \left(\varphi \frac{\partial}{\partial n} \frac{1}{|x|} - \frac{1}{|x|} \frac{\partial \varphi}{\partial n} \right) dS$$

$$- \iint_{\partial B(0,\varepsilon)} \left(\varphi \frac{\partial}{\partial r} \frac{1}{r} - \frac{1}{r} \frac{\partial \varphi}{\partial r} \right) dS$$

$$\Delta \frac{1}{|x|} = 0$$

$$- \iint_{\partial B(0,\varepsilon)} \left(\quad \right) dS = \frac{1}{\varepsilon^2} \iint \varphi \, dS + \frac{1}{\varepsilon} \iint \frac{\partial \varphi}{\partial n} \, dS$$

$$= 4\pi \overline{\varphi} + 4\pi \varepsilon \overline{\frac{\partial \varphi}{\partial r}} \to 4\pi \varphi(0)$$

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