A-1. Let $c$ be any complex number. Show that $\lim_{n \to \infty} \frac{c^n}{n!} = 0$.

A-2. Show that $\sin x$ is not a polynomial.

A-3. Let $A$ be a matrix, not necessarily square. Say $V$ and $W$ are particular solutions of the equations $AV = Y_1$ and $AW = Y_2$, respectively, while $Z \neq 0$ is a solution of the homogeneous equation $AZ = 0$. Answer the following in terms of $V$, $W$, and $Z$.

a) Find some solution of $AX = 3Y_1 - 5Y_2$.

b) Find another solution of $AX = 3Y_1 - 5Y_2$.

c) If $A$ is a square matrix, then $\det A = ?$
A-4. Let $A : \mathbb{R}^3 \to \mathbb{R}^2$ and $B : \mathbb{R}^2 \to \mathbb{R}^3$, so $BA : \mathbb{R}^3 \to \mathbb{R}^3$. Show that $BA$ can not be invertible.

A-5. Let a smooth function $g(x)$ have the three properties: $g(0) = 2$ $g(1) = 0$ $g(4) = 6$. Show that at some point $0 < c < 4$ one has $g''(c) > 0$. To be more specific, find a number $m > 0$ so that $g''(c) \geq m > 0$.

**Part B: Standard problems 60 points (10 points each)**

B-1. Say you have $k$ linear algebraic equations in $n$ variables; in matrix form we write $AX = Y$. Give a proof or counterexample for each of the following assertions.

a) If $n = k$ there is always at most one solution.

b) If $n > k$ you can always solve $AX = Y$.

c) If $n > k$ the nullspace of $A$ has dimension greater than zero.

d) If $n < k$ then for some $Y$ there is no solution of $AX = Y$.

e) If $n < k$ the only solution of $AX = 0$ is $X = 0$. 

B-2. Let $c_n$ be a sequence of real numbers that converges to $C$. Show that their “average” $S_n := \frac{c_1 + c_2 + \cdots + c_n}{n}$ also converges to $C$.

B-3. Compute $\int\int_{\mathbb{R}^2} \frac{1}{[1 + (2x - y + 1)^2 + (x + y + 3)^2]^2} \, dx \, dy$. 
B-4. Is \( k(x) = \sqrt{x} \) uniformly continuous in the set \( \{x \geq 0\} \)? Justify your assertions.

B-5. If the sequence \( \{a_n\} \) is bounded and \( c > 1 \), show that the series \( \sum_{n=1}^{\infty} \frac{a_n}{n^x} \) converges absolutely and uniformly in the interval \( c \leq x < \infty \).
B-6. Let \( u(x, y, t) \) be a solution of the heat equation \( u_t = \Delta u \) for \((x, y)\) in a smoothly bounded open set \( D \subset \mathbb{R}^2 \) and \( t \geq 0 \). Assume that the temperature \( u(x, y, t) = 0 \) for all points \((x, y)\) on the boundary \( B \) of \( D \) for all \( t \geq 0 \).

a) Let \( E(t) := \frac{1}{2} \int \int_D u^2(x, y, t) \, dx \, dy \). Show that \( \frac{dE}{dt} \leq 0 \).

b) Use this to show that with these zero boundary conditions, if the initial temperature is zero, \( u(x, y, 0) = 0 \) for all \((x, y) \in D\), then \( u(x, y, t) = 0 \) for all \( t \geq 0 \).