Calculus Problems

Math 504 – 505 Jerry L. Kazdan

1. Sketch the points (x,y) in the plane $\mathbb{R}^2$ that satisfy $|y-x| \le 2$.

2. A certain function $f(x)$ has the property that $\int_0^x f(t)\,dt = e^x \cos x + C$. Find both f and the constant C.

3. Compute $\lim_{x\to 0} \left(\dfrac{\cos x}{\cos 2x}\right)^{1/x^2}$.

4. Sketch the curve that is defined implicitly by $x^3 + y^3 - 3xy = 0$. Calculate $y'(0)$.

5. Calculate $\sum_{n=1}^{\infty} \dfrac{1}{4n^2-1}$.

6. Determine the indefinite integral $\int \log(1+x^2)\,dx$.

7. Let $f(x)$ be a smooth function for $0 \le x \le 1$. If $f'(x) = 0$ for all $0 \le x \le 1$, what can you conclude? Prove all your assertions.

8. Solve the initial value problem $(1+e^x)yy' = e^x$ with $y(1) = 1$.

9. Let the continuous function $f(\theta)$, $0 \le \theta \le 2\pi$ represent the temperature along the equator at a certain moment, say measured from the longitude at Greenwich.. Show there are antipodal points with the *same* temperature.

10. a) Let $g(x) := x^3(1-x)$. Without computation, show that $g'''(c) = 0$ for some $0 < c < 1$.

    b) Let $h(x) := x^3(1-x)^3$. Show that $h'''(x)$ has exactly three distinct roots in the interval $0 < x < 1$.

    c) Let $p(x) := \left(\dfrac{d}{dx}\right)^4 (1-x^2)^4$. Show that p is a polynomial of degree 4 and that it has 4 real distinct zeroes, all lying in the interval $-1 < x < 1$.

11. In $\mathbb{R}^2$, let $Q_1 = (x_1, y_1)$, $Q_2 = (x_2, y_2)$, and $Q_3 = (x_3, y_3)$, where $x_1 < x_2 < x_3$.

a) Show there is a unique quadratic polynomial $p(x)$ that passes through these points: $p(x_j) = y_j$, $j = 1, 2, 3$.

b) If $y_1 > y_2$ and $y_3 > y_2$ and $f(x)$ is any smooth function that passes through these three points, show there is some point $c \in (x_1, x_3)$ where $f''(c) > 0$. Even better, for some c, $f''(c) \geq p''$, so p'' is the optimal constant. [Remark: It is enough to consider the special case where $x_2 = 0$ and $y_2 = 0$. Then write $x_1 = -a < 0$, $x_3 = b > 0$].

12. a) If $f(x) > 0$ is continuous for all $x \geq 0$ and the improper integral $\int_0^\infty f(x)\,dx$ exists, then $\lim_{x\to\infty} f(x) = 0$. Proof or counterexample.

b) If $f(x) > 0$ is continuous for all $x \geq 0$ and the improper integral $\int_0^\infty f(x)\,dx$ exists, then $f(x)$ is bounded. Proof or counterexample.

13. Find *explicit* rational functions $f(x)$ and $g(x)$ with the following Taylor series: $f(x) = \sum_1^\infty nx^n$, $g(x) = \sum_1^\infty n^2 x^n$.

14. a) Let $x = (x_1, x_2)$ be a point in $\mathbb{R}^2$ and consider $\displaystyle\int_{\mathbb{R}^2} \frac{1}{(1 + \|x\|^2)^p}\,dx$. For which p does this improper integral converge?

b) This integral can be computed explicitly. Do so.

c) Repeat part a) where $x \in \mathbb{R}^3$ and the integral is over $\mathbb{R}^3$ instead of $\mathbb{R}^2$.

15. Compute $\displaystyle\iint_{\mathbb{R}^2} \frac{1}{[1 + (2x + y + 1)^2 + (x - y + 3)^2]^2}\,dx\,dy.$

16. Let $v(x,t) := \displaystyle\int_{x-2t}^{x+2t} g(s)\,ds$, where g is a continuous function. Compute $\partial v/\partial t$ and $\partial v/\partial x$.

17. Let $H(t) := \displaystyle\int_{a(t)}^{b(t)} f(x,t)\,dx$, where $a(t)$, $b(t)$, and $f(x,t)$ are smooth functions of their variables. Compute dH/dt.

18. a) Let $p(x) := x^3 + cx + d$, where c, and d are real. Under what conditions on c and d does this has three distinct real roots? [ANSWER: $c < 0$ and $d^2 < -4c^3/27$].

b) Generalize to the real polynomial $p(x) := ax^3 + bx^2 + cx + d$ $(a \neq 0)$ by a change of variable reducing to the above special case.

19. If $b \geq 0$, show that for every real c the equation $x^5 + bx + c = 0$ has exactly one real root. What if $b < 0$? Say as much as you can.

20. Let $f(t)$ and $g(t)$ be smooth increasing functions of $t \in \mathbb{R}$. Proof or counterexample:

    a) $f(t) + g(t)$ is an increasing functions of t.

    b) $f(t)g(t)$ is an increasing functions of t.

    c) If $f(t) > 0$ and $g(t) > 1$ then $f(t)^{g(t)}$ is an increasing functions of t.

21. Let a smooth function $g(x)$ have the three properties: $g(0) = 2$ $g(1) = 0$ $g(4) = 6$. Show that at some point $0 < c < 4$ one has $g''(c) > 0$. Better yet, find a number $m > 0$ so that $g''(c) \geq m > 0$.

    Is it true that g'' must be positive at at least one point $0 < c < 1$? Proof or counterexample.

22. Let $f(x)$ be a differentiable function for all real x with the property that $f'(x) < 1$ for all x. Show has at most one *fixed point*, that is, at most one point c where $f(c) = c$.

23. Let g be a differentiable function with the properties $g(a) = 0$, $g(b) = 0$, and $g'(x) \geq 0$ for all $x \in [a, b]$. What can you deduce about $g(x)$ for $x \in [a, b]$? Justify your conclusions.

24. Let $v(x)$ be a smooth real-valued function for $0 \leq x \leq 1$. If $v(0) = v(1) = 0$ and $v''(x) > 0$ for all $0 \leq x \leq 1$, show that $v(x) \leq 0$ for all $0 \leq x \leq 1$.

25. If a smooth curve $y = f(x)$ has the property that $f''(x) > 0$, show that the chord joining two points of the curve lies above the curve:

$$tf(b) + (1-t)f(a) \geq f(tb + (1-t)a) \qquad \text{for all} \quad 0 \leq t \leq 1.$$

26. a) Find an integer N so that $\quad 1 + \dfrac{1}{\sqrt{2}} + \dfrac{1}{\sqrt{3}} + \cdots + \dfrac{1}{\sqrt{N}} > 100.$

    b) Find an integer N so that $\quad 1 + \dfrac{1}{2} + \dfrac{1}{3} + \cdots + \dfrac{1}{N} > 100.$

27. Let c be any complex number. Show that $\lim_{n\to\infty} \frac{c^n}{n!} = 0$.

28. a) Show that $\sin x$ is not a polynomial.

    b) Show that $\sin x$ is not a rational function, that is, it cannot be the quotient of two polynomials.

    c) Let $f(t)$ be periodic with period 1, so $f(t+1) = f(t)$ for all real t. If f is not a constant, show that it cannot be a rational function. that is, f cannot be the quotient of two polynomials.

    d) Show that e^x is not a rational function.

29. Let $f(x)$ be a differentiable function of $x := (x_1, x_2, x_3)$ for all $x \in \mathbb{R}^3$. If f is *homogeneous of degree* k in the sense that $f(cx) = c^k f(x)$ for all $c > 0$, show that $x \cdot \nabla f(x) = kf(x)$ (Euler).

30. The *Gamma function* is defined by $\Gamma(x) := \int_0^\infty e^{-t} t^{x-1}\, dt$.

    a) For which real x does this improper integral converge?

    b) Show that $\Gamma(x+1) = x\Gamma(x)$ and deduce that $\Gamma(n+1) = n!$ for any integer $n \geq 0$.

31. Say $\gamma(t) : \mathbb{R} \to \mathbb{R}^2$ defines a smooth curve in the plane.

    a) If $\gamma(0) = 0$ and $\|\gamma'(t)\| \leq c$, show that for any $T \geq 0$, $\|\gamma(T)\| \leq cT$. Moreover, show that equality can occur if and only if one has $\gamma(t) = cvt$ where v is a unit vector that does not depend on t.

    b) If $\gamma(0) = 0$, $\gamma'(0) = 0$ and $\|\gamma''(t)\| \leq 12$, give an upper bound estimate for $\|\gamma(2)\|$. When can this upper bound be achieved?

32. Let $\mathbf{r}(t)$ define a smooth curve that does not pass through the origin.

    a) If the point $\mathbf{a} = \mathbf{r}(t_0)$ is a point on the curve that is closest to the origin (and *not* an end point of the curve), show that the position vector $\mathbf{r}(t_0)$ is perpendicular to the tangent vector $\mathbf{r}'(t_0)$.

    b) What can you say about a point $\mathbf{b} = \mathbf{r}(t_1)$ that is *furthest* from the origin?

33. Consider two smooth plane curves $\gamma_1, \gamma_2 \colon (0,1) \to \mathbb{R}^2$ that do not intersect. Suppose P_1 and P_2 are points on γ_1 and γ_2, respectively, such that the distance $|P_1P_2|$ is minimal. Prove that the straight line P_1P_2 is normal to *both* curves.

34. Let $h(x,y,z) = 0$ define a smooth surface in $\mathbb{R}^3$ and let $P := (a,b,c)$ be a point *not* on the surface. If $Q : (x,y,z)$ is a point on the surface that is closest to P, show that the line PQ is perpendicular to the tangent plane to the surface at Q.

35. Let $\mathbf{r}(t)$ describe a smooth curve and let $\mathbf{V}$ be a fixed vector. If $\mathbf{r}'(t)$ is perpendicular to $\mathbf{V}$ for all t and if $\mathbf{r}(0)$ is perpendicular to $\mathbf{V}$, show that $\mathbf{r}(t)$ is perpendicular to $\mathbf{V}$ for all t.

36. Let $f(s)$ be any differentiable function of the real variable s. Show that $u(x,t) := f(x+3t)$ has the property that $u_t = 3u_x$. Show that u also satisfies the wave equation $u_{tt} = 9u_{xx}$.

37. Let $u(x,y)$ be a smooth function.

   a) If $u_x = 0$ with $u(0,y) = \sin(3y)$, find $u(x,y)$.

   b) If $u_x = 2xy$ with $u(0,y) = \sin(3y)$, find $u(x,y)$.

   c) If $u_x + u_y = 0$ with $u(0,y) = \sin(3y)$, find $u(x,y)$. Is there more than one such function?

   d) If $u_x + u_y = 3 - 2xy$ with $u(0,y) = \sin(3y)$, find $u(x,y)$. Is

   e) If $u_x - 2u_y = 0$ with $u(0,y) = \sin(3y)$, find $u(x,y)$. Is there more than one such function?

38. Let $\mathbf{r} := x\mathbf{i} + y\mathbf{j}$ and $\mathbf{V}(x,y) := p(x,y)\mathbf{i} + q(x,y)\mathbf{j}$ be (smooth) vector fields and C a smooth curve in the plane. In this problem I is the line integral $I = \int_C \mathbf{V} \cdot d\mathbf{r}$. For each of the following, either give a proof or give a counterexample.

   a) If C is a vertical line segment and $q(x,y) = 0$, then $I = 0$.

   b) If C is a circle and $q(x,y) = 0$, then $I = 0$.

   c) If C is a circle centered at the origin and $p(x,y) = -q(x,y)$, then $I = 0$.

   d) If $p(x,y) > 0$ and $q(x,y) > 0$, then $I > 0$.

39. Let C denote the unit circle centered at the origin of the plane, and D denote the circle of radius 5 centered at $(2,1)$, both oriented counterclockwise. Let Q denote the ring region between these curves. If a vector field $\mathbf{V}$ satisfies $\operatorname{div}\mathbf{V} = 0$, show that the line integral $\int_C \mathbf{V} \cdot \mathbf{N}\, ds = \int_D \mathbf{V} \cdot \mathbf{N}\, ds =$ [This extends immediately to the situation where C and D are more general curves and Q is the region between them. For fluid flow it is an expression of *conservation of mass*, since $\operatorname{div}\mathbf{V} = 0$ means there are no sources or sinks in the region Q.]

40. (Integration by Parts for Multiple Integrals) Let $\mathbf{F}$ be a smooth vector field and u a smooth scalar-valued function.

a) Prove the identity $\nabla \cdot (u\mathbf{F}) = \nabla u \cdot \mathbf{F} + u\nabla \cdot \mathbf{F}$. Compare this with the special case of a function of one variable.

b) Let $\mathcal{D}$ be a bounded region in the plane whose boundary is the curve C with unit outer normal $\mathbf{N}$. Also, let u be a scalar-valued function, and $\mathbf{F}$ a vector field. Prove the identity

$$\iint_{\mathcal{D}} u\nabla \cdot \mathbf{F}\, dA = \oint_C u\mathbf{F} \cdot \mathbf{N}\, ds - \iint_{\mathcal{D}} \nabla u \cdot \mathbf{F}\, dA.$$

Notice that for a function of one variable with $\mathcal{D}$ being the interval $\{a < x < b\}$, this reduces precisely to the usual formula for integration by parts.

c) Generalize this formula to the case where $\mathcal{D}$ is a bounded (solid) region in three dimensional space.

d) One frequently uses this with $\mathbf{F} = \nabla v$. Show the above formula for integration by parts becomes (say in two dimensions)

$$\iint_{\mathcal{D}} u\nabla \cdot \nabla v\, dA = \oint_C u\nabla v \cdot \mathbf{N}\, ds - \iint_{\mathcal{D}} \nabla u \cdot \nabla v\, dA.$$

This is *Green's theorem.* To what does this reduce for functions on one variable?

e) As a short application using this, say $u(x,y)$ is a *harmonic function* in a bounded region $\mathcal{D}$, so $\Delta u := \nabla \cdot \nabla u = u_{xx} + u_{yy} = 0$. One can think of $u(x,y)$ as being the equilibrium temperature of $\mathcal{D}$. Let C is the boundary of $\mathcal{D}$. If $u = 0$ on C, it is plausible that one must have $u(x,y) = 0$ throughout $\mathcal{D}$. Show how this follows from the above formula. What is the analogous assertion for functions of one variable, where a harmonic function is just a solution of $u'' = 0$?

41. Let $\mathcal{D}$ be a bounded region in the plane, and let $\mathcal{B}$ be its boundary.

a) Use the divergence theorem (or any related formula you know) to show that for any smooth function $v(x,y)$

$$\iint_{\mathcal{D}} \Delta v\, dx\, dy = \int_{\mathcal{B}} \frac{\partial v}{\partial N}\, ds$$

where $\partial v / \partial N := \nabla v \cdot \mathbf{n}$ is the outer normal directional derivative on $\mathcal{B}$.

b) Let $u(x,y,t)$ be a solution of the heat equation $u_t = \Delta u$ for (x,y) in $\mathcal{D}$. Assume that the boundary, $\mathcal{B}$, is *insulated*, so the outer normal derivative there is zero: $\frac{\partial u}{\partial N} = 0$ on $\mathcal{B}$.

Show that $\quad Q(t) := \iint_{\mathcal{D}} u(x,y,t)\, dx\, dy \quad$ is a constant.

42. Continuing the notation of the previous problem, say that instead the temperature $u(x,y,t)=0$ for all points (x,y) on the boundary $\mathcal{B}$.

a) Show that the function $E(t):=\frac{1}{2}\int_{\mathcal{D}}u^2(x,y,t)\,dx\,dy$ has the property that $dE/dt\le 0$.

b) Use this to show that with these zero boundary conditions, if the initial temperature is zero, $u(x,y,0)=0$, then $u(x,y,t)=0$ for *all* $t\ge 0$.

43. Let $u(x,y,t)$ describe the motion of a vibrating drumhead $\mathcal{D}$. A reasonable mathematical model shows that u satisfies the *wave equation* $u_{tt}=\Delta u$ in $\mathcal{D}$ with *boundary condition* $u(x,y,t)=0$ for all (x,y) on the boundary $\partial\mathcal{D}$.

Physical reasoning leads one to define the *energy* as

$$E(t):=\frac{1}{2}\iint_{\mathcal{D}}(u_t^2+|\nabla u|^2)\,dA.$$

a) Show that energy is conserved: $E(t)=E(0)$. [HINT: Show $dE/dt=0$.]

b) If in addition one knows that the initial position $u(x,y,0)=0$ and that the *initial velocity* $u_t(x,y,0)=0$, show that $E(t)=0$ for all t and deduce that $u(x,y,t)\equiv 0$. [This is hardly a surprise on physical grounds, but it should be interpreted as reassuring us that this mathematical model is indeed reasonably correct.]

44. If $h'(t)\le ch(t)$, where c is a constant, show that $h(t)\le e^{ct}h(0)$ for all $t\ge 0$.

45. Say $u(t)$ satisfies $u''+b(t)u'+c(t)u=0$, where $b(t)$ and $c(t)$ are bounded functions. Let $E(t):=\frac{1}{2}(u'^2+u^2)$.

a) Show that $E'(t)\le\gamma E(t)$, where γ is a constant.

b) Deduce that if $u(0)=0$ and $u'(0)=0$, then $u(t)=0$ for all t.

46. Let $\mathcal{V}:=\{u(x)\in C^2(\mathbb{R})\mid u''+u=0\}$. Prove that $\dim\mathcal{V}=2$. Prove all of your assertions in detail.

47. The solutions to the following matrix differential equation

$$X'=\begin{bmatrix}3&-1\\1&1\end{bmatrix}X$$

form a vector space. Find a basis for this vector space.

48. Consider the differential equation $X'(t) = AX(t)$ where

$$A = \begin{pmatrix} 0 & 0 & -1 \\ 0 & -2 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

Which of the following assertions are correct—and why?

a) There is a solution of the form $X(t) = U$, where U is a real constant (non-zero) vector.

b) There is a solution of the form $X(t) = Ve^{2t}$, where V is a real constant (non-zero) vector.

c) There is a solution of the form $X(t) = Ve^{-2t}$, where V is a real constant (non-zero) vector.

d) There is a complex solution of the form $X(t) = We^{it}$, where W is a constant (non-zero) vector.

e) All of the solutions of this equation remain bounded as $t \to \infty$.

49. Consider the *second order* differential equation $X'' = AX$ where A is a symmetric 2×2 matrix.

a) Find the general solution if $A = \begin{pmatrix} 5 & 0 \\ 0 & -3 \end{pmatrix}$.

b) Find the general solution if $A = \begin{pmatrix} 1 & 4 \\ 4 & 1 \end{pmatrix}$. [Suggestion: First diagonalize A, so $D := R^{-1}AR$ is diagonal. Then make the change of variables $X = RY$ to obtain a simpler differential equation for $Y(t)$.]

c) Find the general solution if $A = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}$.

50. For which complex numbers z does the series $\sum_1^\infty ne^{-nz}$ converge?

51. a) Let $u(x_1, \ldots, x_n)$ be a smooth function that depends only on the distance $r = \sqrt{x_1^2 + \cdots + x_n^2}$. Show that

$$\frac{\partial^2 u}{\partial x_j^2} = \frac{x_j^2}{r^2}\frac{d^2 u}{dr^2} + \frac{(r^2 - x_j^2)}{r^3}\frac{du}{dr}, \quad \text{and hence} \quad \Delta u = u_{rr} + \tfrac{n-1}{r}u_r.$$

b) Find all harmonic functions (these are the solutions of $\Delta u = 0$) that depend only on r.

52. a) Find the equation of the tangent plane to the surface $x^2 + xy + y^3 - z^2 = 2$ at the point $(1, 1, 1)$.

    b) Say the function $T = x^2 + xy + y^3 - z^2$ gives the temperature at the point (x, y, z). At the point $(1, 1, 1)$, in which direction should one move so that the temperature increases fastest?

53. Let $\psi(t)$ be a scalar-valued function with a continuous derivative for $0 < t < \infty$ and let $\mathbf{X} = (x, y, z) \in \mathbb{R}^3$. Define the vector field $\mathbf{F}(\mathbf{X}) := \psi(\|\mathbf{X}\|)\mathbf{X}$ for all $\mathbf{X} \neq 0$. Show that this vector field is conservative by finding a scalar-valued function $\varphi(r)$ with the property that $\mathbf{F}(\mathbf{X}) := \nabla\varphi(\|\mathbf{X}\|)$. In particular, this shows that *every central force field is conservative.*

54. Let $\mathcal{D}$ be a bounded region in the plane with smooth boundary $\mathcal{B}$. Show that

$$\text{Area}(\mathcal{D}) = \tfrac{1}{2}\int_{\mathcal{B}} x\,dy - y\,dx.$$

Use this to find the area inside the ellipse $(x, y) = (a\cos\theta, b\sin\theta)$ for $0 \le \theta \le 2\pi$.

55. If $\{b_j\} > 0$, prove the arithmetic-geometric mean inequality

$$(b_1 b_2 \cdots b_n)^{1/n} \le \frac{b_1 + b_2 + \cdots + b_n}{n}.$$

When is there equality?

56. Let $0 < c < 1$. Show that $s^c t^{1-c} < cs + (1-c)t$ for all $s, t > 0$, $s \neq t$ (if $s = t$, then this becomes an equality).

57. Let $p, q \ge 1$ with $\dfrac{1}{p} + \dfrac{1}{q} = 1$. Show that $xy \le \dfrac{x^p}{p} + \dfrac{y^q}{q}$ for all $x, y > 0$.

58. Let $P_1, \ldots P_n$ be $n \ge 3$ points on a circle and let Q be the polygon obtained by connecting these successive points. How should the points be situated to maximize the area of Q?

59. a) Find a smooth function $f:\mathbb{R}^2 \to \mathbb{R}$ that has exactly three critical points, all non-degenerate, one being a local max, one a local min, and the third a saddle.

b) Show there is no such $f(x,y)$ of the form $f(x,y) = g(x) + h(y)$.

60. Compute $\lim_{\lambda\to\infty} \int_0^1 |\sin \lambda x|\, dx$ (part of the problem is to show that the limit exists).

61. a) State what it means for a real-valued function defined on the closed, bounded interval $[a,b]$ to be Riemann integrable.

b) Using your definition from part (a), prove that any monotonically increasing function on [0,1] is Riemann integrable.

62. Given the vector field $\mathbf{V}(x,y,z) = (4y, x, 2z)$ in 3-space, find the value of the integral

$$\iint_H \operatorname{curl} \mathbf{V} \cdot \mathbf{n}\, dA$$

where H is the hemisphere $x^2 + y^2 + z^2 = a^2$, $z \geq 0$, $\mathbf{n}$ is the unit outward normal, and dA is the element of area.

63. a) Let $c(x)$ be a given smooth function and $u(x) \not\equiv 0$ satisfy the differential equation $-u'' + c(x)u = \lambda u$ on the bounded interval $\Omega = \{a < x < b\}$ with $u = 0$ on the boundary of Ω. Show that

$$\lambda = \frac{\int_\Omega (u'^2 + cu^2)\, dx}{\int_\Omega u^2\, dx}$$

b) Let $c(x,y)$ be a given smooth function and $u(x,y) \not\equiv 0$ satisfy the differential equation $-(u_{xx} + u_{yy}) + cu = \lambda u$ on a bounded set $\Omega \subset \mathbb{R}^2$ with $u = 0$ on the boundary of Ω. Show that

$$\lambda = \frac{\iint_\Omega (|\nabla u|^2 + cu^2)\, dx\, dy}{\iint_\Omega u^2\, dx\, dy}$$

64. Investigate the continuity and differentiability of

$$f(x) = \begin{cases} |x|^p \cos \frac{1}{x} & \text{for } x \neq 0, \\ 0 & \text{for } x = 0, \end{cases},$$

where p is a real number.

65. Determine the radius of convergence of the series $\displaystyle\sum_{n=0}^{\infty} \frac{x^{n^2}}{n!}$.

66. Calculate $\displaystyle\lim_{x\to\infty} \{x^{1/3}[(x+1)^{2/3}-(x-1)^{2/3}]\}$.

67. Prove that the function $f(x) = \left(1+\dfrac{1}{x}\right)^x$ is monotone increasing for $x>0$.

68. Compute $\displaystyle\int \frac{dx}{\sin x + \cos x}$.

69. Find the critical points of each of the following functions defined on the plane $\mathbb{R}^2$. Also, where possible, classify these critical points as local maxima, minima, or saddles.

a) $f(x,y) = x^4 + y^4 - 4xy + 1$

b) $g(x,y) = x^2y^2$

c) $\dfrac{\cos x}{1+y^2}$

70. Find an example of a smooth function $f(x,y)$ defined on the whole plane $\mathbb{R}^2$ that has exactly three critical points, all non-degenerate, with one a local maximum, one a local minimum, and the third a saddle point.

71. Here we use the series $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$ to show that e is irrational.

a) Show that $2 < e < 3$, so e is not an integer.

b) Assume $e = p/q$ is rational, with p and q integers with $q \geq 2$. Use Taylor series with q terms and remainder R_q to show that $e \cdot q! = N + \frac{e^c}{q+1}$, where N is an integer and $0 < c < 1$.

c) Deduce that $\frac{e^c}{q+1}$ is an integer. Show this contradicts $e^c < e^1 < 3$ and $q+1 \geq 3$.

72. Let $h(x) \geq 0$ be a continuous monotonically decreasing function for $0 \leq x \leq \infty$ with the property that $\lim_{x\to\infty} h(x) = 0$. Show that the improper integral $\int_0^\infty h(x) \sin x \, dx$ exists.

73. Let $f \in C^2(a,b)$ and say $x_0 \in (a,b)$. If $f''(x_0) > 0$, give an analytic proof that near x_0 the graph of $y = f(x)$ lies above its tangent line at $(x_0, f(x_0))$.

74. Let $I_k = \int_{-\infty}^{\infty} x^{2k} e^{-x^2}\, dx, \quad k \geq 0.$

a) Show that $I_k = \frac{2k-1}{2} I_{k-2}$.

b) Compute I_2, I_3, I_4, I_5, I_6, and I_7. [You may use that $\int_{-\infty}^{\infty} e^{-x^2}\, dx = \sqrt{\pi}$].

[Last revised: January 25, 2013]