Directions This exam has three parts, Part A has 4 problems asking for Examples (20 points, 5 points each), Part B asks you to describe some sets (20 points), Part C has 4 traditional problems (60 points, 15 points each). Closed book, no calculators – but you may use one 3” × 5” card with notes.

Part A: Examples (4 problems, 5 points each). Give an example of an infinite set in a metric space (perhaps \( \mathbb{R} \)) with the specified property.

A–1. Bounded with exactly two limit points.

A–2. Containing all of its limit points.

A–3. Distinct points \( \{x_j\}, j = 1, 2, \ldots \) with \( x_i \neq x_j \) for \( i \neq j \) that is compact.


Part B: Classify sets (20 points) For each of the following sets, circle the listed properties it has:

a) \( \{1 + \frac{1}{n} \in \mathbb{R}, \ n = 1, 2, 3, \ldots \} \) open closed bounded compact countable

b) \( \{1\} \cup \{1 + \frac{1}{n} \in \mathbb{R}, \ n = 1, 2, 3, \ldots \} \)

open closed bounded compact countable

c) \( \{(x, y) \in \mathbb{R}^2 : 0 < y \leq 1\} \) open closed bounded compact countable

d) \( \{(x, y) \in \mathbb{R}^2 : x = 0\} \) open closed bounded compact countable

e) \( \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\} \) open closed bounded compact countable

f) \( \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\} \) open closed bounded compact countable

g) \( \{(x, y) \in \mathbb{R}^2 : y > x^2\} \) open closed bounded compact countable

h) \( \{(k, n) \in \mathbb{R}^2 : k, n \text{ any positive integers}\} \)

open closed bounded compact countable
**Part C: Traditional Problems** (4 problems, 20 points each)

C-1. In $\mathbb{R}$, if $a_n \to A$ and $b_n \to B$, show that the product $a_nb_n \to AB$.

C-2. Given a real sequence $\{a_k\}$, let $C_n = \frac{a_1 + \cdots + a_n}{n}$ be the sequence of averages (arithmetic mean). If $a_k$ converges to $A$, show that the averages $C_n$ also converge to $A$.

C-3. Let $K_j$, $j = 1, 2, \ldots$ be compact sets in a metric space. Give a proof or counterexample for each of the following assertions.
   a) $K_1 \cap K_2$ is compact.
   b) $K_1 \cup K_2$ is compact.
   c) $\bigcup_{j=1}^{\infty} K_j$ is compact.

C-4. In a complete metric space $M$, let $d(x, y)$ denote the distance. Assume there is a constant $0 < c < 1$ so that the sequence $x_k$ satisfies $$d(x_{n+1}, x_n) < cd(x_n, x_{n-1}) \text{ for all } n = 1, 2, \ldots.$$  

   a) Show that $d(x_{n+1}, x_n) < c^n d(x_1, x_0)$.
   b) Show that the $\{x_k\}$ is a Cauchy sequence.
   c) Show that there is some $p \in M$ so that $\lim_{n \to \infty} x_k = p$. 