Directions  This exam has three parts, Part A has 4 problems asking for Examples (20 points, 5 points each), Part B asks you to describe some sets (20 points), Part C has 4 traditional problems (60 points, 15 points each).
Closed book, no calculators – but you may use one 3" × 5" card with notes.

Part A: Examples (4 problems, 5 points each). Give an example of an infinite set in a metric space (perhaps \( \mathbb{R} \)) with the specified property.

A–1. Bounded with exactly two limit points.

Solution: The set \( \{(-1)^n(1 + \frac{1}{n}), \; n = 1, 2, 3, \ldots \} \) in \( \mathbb{R} \).

A–2. Containing all of its limit points.

Solution: Lots of examples: 1). The empty set. 2). All of \( \mathbb{R} \). 3). The point \( \{0\} \in \mathbb{R} \). 4). The closed interval \( \{0 \leq x \leq 1 \in \mathbb{R} \} \).

A–3. Distinct points \( \{x_j, \; j = 1, 2, 3, \ldots \} \) with \( x_i \neq x_j \) for \( i \neq j \) that is compact.

Solution: The following subset of the real numbers: \( \{0\} \cup \{\frac{1}{n}, \; n = 1, 2, 3, \ldots \} \).


Solution: The closed unit ball \( \|x\| \leq 1 \) in \( \ell_2 \). The standard basis vectors \( e_1 = (1, 0, 0, \ldots), e_2 = (0, 1, 0, 0, \ldots), \) etc have no convergent subsequence.
Another example: the real numbers \( \{x \in \mathbb{R} \mid 0 \leq x \leq 1 \} \) with the discrete metric: \( d(x, y) = 1 \) for \( x \neq y, \; d(x, x) = 0 \).

Part B: Classify sets (20 points) For each of the following sets, circle the listed properties it has:

a) \( \{1 + \frac{1}{n} \in \mathbb{R}, \; n = 1, 2, 3, \ldots \} \) open closed \( \text{bounded} \) compact \( \text{countable} \)

b) \( \{1\} \cup \{1 + \frac{1}{n} \in \mathbb{R}, \; n = 1, 2, 3, \ldots \} \)

open \( \text{closed} \) \( \text{bounded} \) \( \text{compact} \) \( \text{countable} \)

c) \( \{(x, y) \in \mathbb{R}^2 : 0 < y \leq 1 \} \) open closed \( \text{bounded} \) compact \( \text{countable} \)

d) \( \{(x, y) \in \mathbb{R}^2 : x = 0 \} \) open \( \text{closed} \) \( \text{bounded} \) compact \( \text{countable} \)
e) \{ (x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1 \} \quad \text{open} \quad \text{closed} \quad \text{bounded} \quad \text{compact} \quad \text{countable}

f) \{ (x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1 \} \quad \text{open} \quad \text{closed} \quad \text{bounded} \quad \text{compact} \quad \text{countable}

g) \{ (x, y) \in \mathbb{R}^2 : y > x^2 \} \quad \text{open} \quad \text{closed} \quad \text{bounded} \quad \text{compact} \quad \text{countable}

h) \{ (k, n) \in \mathbb{R}^2 : k, n \text{ any positive integers} \}

Part C: Traditional Problems (4 problems, 20 points each)

C-1. In \( \mathbb{R} \), if \( a_n \to A \) and \( b_n \to B \), show that the product \( a_n b_n \to AB \).

Solution: Let \( p_n = a_n - A \to 0 \), \( q_n = b_n - B \to 0 \). Then
\[
a_n b_n = (p_n + A)(q_n + B) = p_n q_n + Aq_n + Bp_n + AB.
\]

Using that for convergent sequences \( x_n \) and \( y_n \) we know \( \lim(x_n + y_n) = \lim x_n + \lim y_n \) and \( \lim(cx_n) = c \lim x_n \), we see that it is enough to show that \( p_n q_n \to 0 \). Given \( \epsilon > 0 \) (which we may assume satisfies \( \epsilon < 1 \)), pick \( N \) so that if \( n > N \) then \( |p_n| < \epsilon \) and \( |q_n| < \epsilon \). Consequently \( |p_n q_n| < \epsilon^2 < \epsilon \).

C-2. Given a real sequence \( \{a_k\} \), let \( C_n = \frac{a_1 + \cdots + a_n}{n} \) be the sequence of averages (arithmetic mean). If \( a_k \) converges to \( A \), show that the averages \( C_n \) also converge to \( A \).

Solution: Letting \( B_n = a_n - A \to 0 \), I could reduce to the case \( A = 0 \). Instead, for variety I proceed directly. Note that
\[
C_n - A = \frac{a_1 + \cdots + a_n}{n} - A = \frac{(a_1 - A) + \cdots + (a_n - A)}{n}
\]

Given any \( \epsilon > 0 \), pick \( N \) so that if \( n > N \) then \( |a_n - A| < \epsilon \). Then write
\[
C_n - A = \frac{(a_1 - A) + \cdots + (a_N - A)}{n} + \frac{(a_{N+1} - A) + \cdots + (a_n - A)}{n}
\]

Now
\[
|J_n| < \frac{n - (N + 1)}{n} \epsilon \leq \frac{n\epsilon}{n} = \epsilon \quad \text{for any} \quad n > N.
\]

We will show that by choosing \( n \) even larger, we can make \( |I_n| < \epsilon \). Since the sequence \( a_n - A \) converges, it is bounded, so for some \( M \) we have \( |a_n - A| < M \). Thus for \( n \) sufficiently large
\[
|I_n| < \frac{NM}{n} < \epsilon.
\]

Consequently, \( |C_n - A| \leq |I_n| + |J_n| < 2\epsilon \).
C–3. Let $K_j$, $j = 1, 2, \ldots$ be compact sets in a metric space. Give a proof or counterexample for each of the following assertions.

a) $K_1 \cap K_2$ is compact.

**Solution:** True. Since compact sets are closed, then $K_1 \cap K_2$ is a closed subset of the compact set $K_1$, and hence compact.

b) $K_1 \cup K_2$ is compact.

**Solution:** True. Let $\{U_\alpha\}$ be any open cover of $K_1 \cup K_2$. A finite number of these, say $\{V_1, \ldots, V_k\}$, cover $K_1$, and $\{W_1, \ldots, W_n\}$, cover $K_2$. Then $\{V_1 \cup \ldots \cup V_k \cup W_1 \cup \ldots \cup W_n\}$ is the desired finite cover of $K_1 \cup K_2$.

c) $\bigcup_{j=1}^\infty K_j$ is compact.

**Solution:** Counterexample. The non-negative real numbers $\{x \geq 0\}$ is the union of the compact sets (closed intervals) $K_j = \{j - 1 \leq x \leq j; j = 1, 2, \ldots\}$. Since this set is not bounded, it is not compact.

C–4. In a complete metric space $M$, let $d(x, y)$ denote the distance. Assume there is a constant $0 < c < 1$ so that the sequence $x_k$ satisfies

$$d(x_{n+1}, x_n) < cd(x_n, x_{n-1})$$

for all $n = 1, 2, \ldots$.

a) Show that $d(x_{n+1}, x_n) < c^n d(x_1, x_0)$.

**Solution:** Since $d(x_2, x_1) < cd(x_1, x_0)$, then

$$d(x_3, x_2) < cd(x_2, x_1) < c^2 d(x_1, x_0).$$

Using this,

$$d(x_4, x_3) < cd(x_3, x_2) < c^3 d(x_1, x_0).$$

The induction to the general case is obvious.

b) Show that the $\{x_k\}$ is a Cauchy sequence.

**Solution:** Say $n > k$. Then using the previous part and that $0 < c < 1$

$$d(x_n, x_k) \leq d(x_n, x_{n-1}) + \ldots + d(x_{k+1}, x_k)$$

$$\leq (c^{n-1} + c^{n-2} + \ldots + c^k) d(x_1, x_0)$$

$$\leq (c^k (1 + c + c^2 + c^3 + \ldots) d(x_1, x_0) = \frac{c^k}{1-c} d(x_1, x_0).$$

Pick $N$ so that $c^N < \epsilon$. If $n > k > N$ then

$$d(x_n, x_k) \leq \frac{\epsilon}{1-c} d(x_1, x_0).$$

c) Show that there is some $p \in M$ so that $\lim_{n \to \infty} x_k = p$.

**Solution:** Since the metric space is complete, there is a point $p$ in the metric space to which the Cauchy sequence $x_k$ converges.