Problem Set 2
DUE: Tues. Sept. 26, 2006. Late papers accepted until 1:00 Wednesday.

Math 508, Fall 2006                Jerry L. Kazdan

1. (Rudin, p.43 #2) A complex number algebraic if it is a root of a polynomial $a_0z^n + \cdots + a_n$ whose coefficients are all integers. Prove that the set of all algebraic numbers is countable. [HINT: For every positive integer $N$ there are only finitely many equations with $n + |a_0| + \cdots + |a_n| = N$.

2. (Rudin, p.43 #5) Construct a bounded set of real numbers with exactly three limit points.

3. (Rudin, p.43 #6) Let $E'$ be the set of limit points of a set $E$ in a metric space. Show that $E'$ is closed.

4. (Rudin, p.43 #10) Let $X$ be any infinite set and for $p, q \in X$ define the function

$$d(p, q) = \begin{cases} 1 & \text{if } p \neq q, \\ 0 & \text{if } p = q \end{cases}$$

Prove that this is a metric (although it is not very interesting). Which subsets are open? closed? compact?

5. (Rudin, p.45 #22) If $x$ and $y$ are real numbers, define

$$d_1(x, y) = (x - y)^2; \quad d_2(x, y) = \sqrt{|x - y|}; \quad d_3(x, y) = |x^2 - y^2|;$$

$$d_4 = |x - 2y|; \quad d_5 = \frac{|x - y|}{1 + |x - y|}.$$  

Which of these define metrics? Justify your assertions.

6. (Rudin, p.44 #20) Are the closures and interiors of connected sets always connected? [Look at subsets of $\mathbb{R}^2$.]

7. (Rudin, p.45 #22) A metric space is called separable if it contains a countable dense subset. Show that $\mathbb{R}^2$ is separable. [HINT: Consider the set of points whose coordinates are rational.]
8. Define two real numbers $x$ and $y$ to be equal if $|x - y|$ is an integer, thus we have a topological circle whose circumference is one.

Let $\alpha$ be an irrational real number, $0 < \alpha < 1$ and consider its integer multiples, $\alpha$, $2\alpha$, $3\alpha$ . . . . Show that this set is dense in $0 \leq x \leq 1$.

[Last revised: September 24, 2006]