Homework Set 5
DUE: Thurs. Oct. 19, 2006. Late papers accepted until 1:00 Friday.

Math 508, Fall 2006  Jerry Kazdan

1. Let \( V \) and \( W \) be normed linear spaces and \( L : V \to W \) a linear map, so \( L(X + Y) = LX + LY \) and \( L(cX) = cLX \). Define the norm of \( L \) by

\[
\|L\| := \sup_{X \neq 0} \frac{\|LX\|_W}{\|X\|_V}.
\]

We say that \( L \) is bounded if \( \|L\| < \infty \)

a) The set \( L(V, W) \) of all linear maps from \( V \) to \( W \) is itself a linear space — since one can add maps, \( L + M \), and multiply them by scalars, \( cL \). Show that \( \|L\| \) defines a norm on \( L(V, W) \), that is,

i). \( \|L\| \geq 0 \), with \( \|L\| = 0 \) only if \( L = 0 \),
ii). \( \|cL\| = |c|\|L\| \) for any scalar \( c \),
iii). \( \|L + M\| \leq \|L\| + \|M\| \) (triangle inequality).

b) Show that \( \|L\| = \sup_{\|X\|_V = 1} \|LX\|_W \).

c) Let \( A = \begin{pmatrix} 0 & 1 \\ 1 & 2 \\ -1 & 0 \end{pmatrix} \) define a linear map from \( \mathbb{R}^3 \) to \( \mathbb{R}^2 \) with the usual euclidean norms. Show that \( A \) is bounded. (You need not compute \( \|A\| \), only get an upper bound for it).

d) If \( L : \mathbb{R}^k \to \mathbb{R}^n \) is given by any \( n \times k \) matrix and both \( \mathbb{R}^k \) and \( \mathbb{R}^n \) have the Euclidean norm, show that \( L \) is bounded.

e) Show that if \( L : \ell_2 \to \ell_2 \) is defined by

\[
LX := (c_1x_1, c_2x_2, c_3x_3, \ldots),
\]

where \( c_j \) is a bounded sequence, then \( L \) is bounded.

f) Show that if \( L : \ell_2 \to \ell_2 \) is defined by

\[
LX := (x_1, 2x_2, \ldots, nx_n, \ldots)
\]

is not a bounded linear map.

2. Let \( f : \mathbb{R} \to \mathbb{R} \) have the property that \( |f(x) - f(\hat{x})| < c|x - \hat{x}| \) for all real \( x, \hat{x} \), and where \( c \) is a constant. Given any starting point \( x_0 \), define \( x_j, j = 1, 2, \ldots \), recursively by the rule

\[
x_{j+1} = f(x_j).
\]

If \( c < 1 \), prove that the \( x_j \) converge to some real number.
3. a) Let $A$ be a subset of the real numbers. Prove that the following statements are equivalent:
   - $A$ is closed.
   - Every sequence $x_n \in A$ such that $\sum_{n=1}^{\infty} \|x_{n+1} - x_n\| < \infty$ converges to a limit in $A$.

   b) Show this is also true for subsets $A$ of the plane $\mathbb{R}^2$ with the usual euclidian norm.

4. Let $\ell_2$ be the usual normed linear space space of infinite sequences $X = (x_1, x_2, \ldots)$ with finite norm: $\|X\| := \sqrt{\sum_{j=1}^{\infty} |x_j|^2} < \infty$. Prove that $\ell_2$ is complete.  
   [SUGGESTION: See the similar proof that $\ell_1$ is complete on the class web page.]