Homework Set 6
DUE: Thurs. Nov. 2, 2006. Late papers accepted until 1:00 Friday.

Math 508, Fall 2006

1. If $L : \ell_2 \to \ell_2$ is defined by $LX := (c_1x_1, c_2x_2, c_3x_3, \ldots)$, where $c_j$ is a bounded sequence, is $L$ is bounded? Proof or counterexample.

2. Show that a linear map $L : V \to W$ between normed vector spaces $V$ and $W$ is continuous at any point $X_0$ if and only if $L$ is continuous at the origin.

3. [CONTINUATION] Show that a linear map $L : V \to W$ is continuous if and only if it is bounded.

4. Let $\mathcal{M}$ and $\mathcal{N}$ be metric spaces and $f : \mathcal{M} \to \mathcal{N}$ be a continuous map. Say $f : p \mapsto q$ and $r \in \mathcal{N}$ with $r \neq q$. Show there is some neighborhood of $p$ whose image does not contain $r$. In other words, there is some open set $U \subset \mathcal{M}$ containing $p$ with the property that $r \not\in f(U)$.

5. Let $f$ be a continuous map from $[0, 1]$ to itself. Show that $f$ has at least one fixed point, that is, a point $c$ so that $f(c) = c$.

6. Show that at any time there are at least two diametrically opposite points on the equator of the earth with the same temperature.

7. [Rudin, p. 98 # 3]. Let $\mathcal{M}$ be a metric space and $f : \mathcal{M} \to \mathbb{R}$ a continuous function. Denote by $Z(f)$ the zero set of $f$. These are the points $p \in \mathcal{M}$ where $f$ is zero, $f(p) = 0$.
   a) Show that $Z(f)$ is a closed set.
   b) [See also Rudin, p. 101 #20] Given any set $E \in \mathcal{M}$, the distance of a point $x$ to $E$ is defined by
      $$h(x) = \rho_E(x) := \inf_{z \in E} d(x, z).$$
      Show that $h$ is a uniformly continuous function.
   c) Use the previous part to show that given any closed set $E \in \mathcal{M}$, there is a continuous function that is zero on $E$ and positive elsewhere.
8. [Rudin, p. 98 # 4]. Let \( f \) and \( g \) be continuous mappings of a metric space \( X \) into a metric space \( Y \) and let \( E \) be a dense subset of \( X \).

   a) Prove that \( f(E) \) is dense in \( f(X) \).

   b) If \( g(p) = f(p) \) for all \( p \in E \), prove that \( g(p) = f(p) \) for all points \( p \) in \( X \). Thus, a continuous function is determined by its values in a dense subset of its domain.

9. [Rudin, p. 99 # 7]. For points \((x, y) \neq (0, 0) \in \mathbb{R}^2\), define

   \[
   f(x, y) = \frac{xy^2}{x^2 + y^4} \quad \text{and} \quad g(x, y) = \frac{xy^2}{x^2 + y^6},
   \]

   while define \( f(0, 0) = 0 \) and \( g(0, 0) = 0 \).

   a) Show that \( f \) is bounded in \( \mathbb{R}^2 \) but not continuous at the origin, while \( g \) is unbounded in every neighborhood of the origin and hence also not continuous there.

   b) Let \( S \in \mathbb{R}^2 \) be any straight line through the origin. Show that if the points \((x, y)\) are strict to lie on \( S \), then both \( f(x, y) \) and \( g(x, y) \) are continuous. MORAL: It can be misleading to understand a function by only examining it on straight lines.

10. [Rudin, p. 99 # 8]. Let \( E \subset \mathbb{R} \) be a set and \( f : E \to \mathbb{R} \) be uniformly continuous.

    a) If \( E \) is a bounded set, show that \( f(E) \) is a bounded set.

    b) If \( E \) is not bounded, give an example showing that \( f(E) \) might not be bounded.

11. [Rudin, p. 99 # 13 or #11] extension by continuity Let \( X \) be a metric space, \( E \subset X \) a dense subset, and \( f : E \to \mathbb{R} \) a uniformly continuous function. Show that \( f \) has a unique continuous extension to all of \( X \). That is, there is a unique continuous function \( g : X \to \mathbb{R} \) with the property that \( g(p) = f(p) \) for all \( p \in X \). [REMARK: One generalize this by replacing \( \mathbb{R} \) by any complete metric space.]

12. [Rudin, p. 101 # 23]. A real-valued function \( f : (a, b) \to \mathbb{R} \) is called convex if

   \[
   f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) \quad \text{for all} \ x, y \in (a, b) \quad \text{and} \quad 0 < t < 1.
   \]

   a) Prove that every convex function is continuous.

   b) Prove that every increasing convex function of a convex function is convex. Example: Assuming \( e^t \) is convex (it is), if \( f \) is convex then so is \( e^{f(x)} \).
13. [Rudin, p. 101 # 24]. [CONTINUATION] Assume that $f : (a,b) \to \mathbb{R}$ is continuous and has the property that

$$f\left(\frac{x+y}{2}\right) \leq \frac{f(x) + f(y)}{2} \quad \text{for all} \quad x, y \in (a,b).$$

Prove that $f$ is convex. [REMARK: One can use this to give a short proof of the arithmetic-geometric mean inequality. Homework Set 3 #10].