Homework Set 7
DUE: Thurs. Nov. 9, 2006. Late papers accepted until 1:00 Friday.

Math 508, Fall 2006

Note: We say a function is smooth if its derivatives of all orders exist and are continuous.

1. Which of the following are uniformly continuous in the set \( \{x \geq 0\} \)? Justify your assertions.
   a). \( f(x) = 2 + 3x \)  
   b). \( g(x) = \sin 2x \)  
   c). \( h(x) = x^2 \)  
   d). \( k(x) = \sqrt{x} \),

2. a) Show that \( \sin x \) is not a polynomial.
   b) Show that \( \sin x \) is not a rational function, that is, it cannot be the quotient of two polynomials.
   c) Let \( f(t) \) be periodic with period 1, so \( f(t + 1) = f(t) \) for all real \( t \). If \( f \) is not a constant, show that it cannot be a rational function. That is, \( f \) cannot be the quotient of two polynomials.
   d) Show that \( e^x \) is not a rational function.

3. a) If a smooth curve \( y = f(x) \) has the property that \( f''(x) \geq 0 \), show that it is convex.
   b) Let \( v(x) \) be a smooth real-valued function for \( 0 \leq x \leq 1 \). If \( v(0) = v(1) = 0 \) and \( v''(x) > 0 \) for all \( 0 \leq x \leq 1 \), show that \( v(x) \leq 0 \) for all \( 0 \leq x \leq 1 \).
   c) Prove that the function \( e^x \) is convex.
   d) Show that \( e^x \geq 1 + x \) for all real \( x \).

4. a) Let \( p(x) := x^3 + cx + d \), where \( c \) and \( d \) are real. Under what conditions on \( c \) and \( d \) does this have three distinct real roots? [Answer: \( c < 0 \) and \( d^2 < -4c^3/27 \)].
   b) Generalize to the real polynomial \( p(x) := ax^3 + bx^2 + cx + d \) \( (a \neq 0) \) by a change of variable \( t = x - \alpha \) (with a clever choice of \( \alpha \)) to reduce to the above special case.

5. Let a smooth function \( g(x) \) have the three properties: \( g(0) = 2 \)  
   \( g(1) = 0 \)  
   \( g(4) = 6 \). Show that at some point \( 0 < c < 4 \) one has \( g''(c) > 0 \). Better yet, find a number \( m > 0 \) so that \( g''(c) \geq m > 0 \).
   Is it true that \( g'' \) must be positive at at least one point \( 0 < c < 1 \) ? Proof or counterexample.
6. Let \( \mathbf{r}(t) \) define a smooth curve that does not pass through the origin.
   a) If the point \( \mathbf{a} = \mathbf{r}(t_0) \) is a point on the curve that is closest to the origin (and not an end point of the curve), show that the position vector \( \mathbf{r}(t_0) \) is perpendicular to the tangent vector \( \mathbf{r}'(t_0) \).
   b) What can you say about a point \( \mathbf{b} = \mathbf{r}(t_1) \) that is furthest from the origin?

7. If \( h : \mathbb{R} \to \mathbb{R} \) is a differentiable function that satisfies \( h'(t) \leq ch(t) \), where \( c \) is a constant, show that \( h(t) \leq e^{ct}h(0) \) for all \( t \geq 0 \).

8. Say \( u(t) \) satisfies \( u'' + b(t)u' + c(t)u = 0 \), where \( b(t) \) and \( c(t) \) are bounded functions. Let \( E(t) := \frac{1}{2}(u'^2 + u^2) \).
   a) Show that \( E'(t) \leq \gamma E(t) \), where \( \gamma \) is a constant.
   b) Use the result of the previous problem to deduce that if \( u(0) = 0 \) and \( u'(0) = 0 \), then \( u(t) = 0 \) for all \( t \).

9. Let \( w(x) \) be a smooth function that satisfies \( w'' - c(x)w = 0 \), where \( c(x) > 0 \) is a given function, show that \( w \) cannot have a local positive maximum (that is, a local maximum where the function is positive). Also show that \( w \) cannot have a local negative minimum.

10. a) For any integer \( n \geq 0 \), show that \( \lim_{x \to 0} \frac{e^{-1/x}}{x^n} = 0 \).
    b) Define \( f : \mathbb{R} \to \mathbb{R} \) by
        \[
        f(x) = \begin{cases} 
        e^{-1/x} & \text{for } x > 0, \\
        0 & \text{for } x \leq 0, 
        \end{cases}
        \]
        Sketch the graph of \( f \).
    c) Show that \( f \) is a smooth function for all real \( x \)
    d) Show that each of the following are smooth and sketch their graphs:
        \[
        g(x) = f(x) - f(1-x) \quad h(x) = \frac{f(x)}{f(x) + f(1-x)} \\
        k(x) = h(x)h(4-x) \quad K(x) = k(x+2), \\
        \phi(x,y) = K(x)K(y), (x,y) \in \mathbb{R}^2 \quad \Phi(x) = K(||x||), x = (x_1,x_2) \in \mathbb{R}^2
        \]