Homework Set 8
DUE: Thurs. Nov. 16, 2006. Late papers accepted until 1:00 Friday.

Math 508, Fall 2006

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Note: We say a function is smooth if its derivatives of all orders exist and are continuous.

1. Let \( r(t) \) describe a smooth curve in \( \mathbb{R}^3 \) and let \( V \) be a fixed vector. If \( r'(t) \) is perpendicular to \( V \) for all \( t \) and if \( r(0) \) is perpendicular to \( V \), show that \( r(t) \) is perpendicular to \( V \) for all \( t \).

2. A diffeomorphism is a smooth invertable map whose inverse map is also smooth.
   a) Find a diffeomorphism \( f : \mathbb{R} \to \mathbb{R}_+ \), where \( \mathbb{R}_+ = \{ x \in \mathbb{R} | x > 0 \} \).
   b) Find a diffeomorphism \( g : \mathbb{R} \to \Omega \), where \( \Omega \) is the interval: \( \Omega = \{ x \in \mathbb{R} | 0 < x < 1 \} \).
   c) Find a diffeomorphism \( F : \mathbb{R}^2 \to \mathbb{R}_+^2 \), where \( \mathbb{R}_+^2 = \{(x,y) \in \mathbb{R}^2 | y > 0 \} \).
   d) Find a diffeomorphism \( G : \mathbb{R}^2 \to \Omega \), where \( \Omega \) is the strip: \( \Omega = \{(x,y) \in \mathbb{R}^2 | 0 < y < 1 \} \).

3. Let \( f(x) \) be a smooth function for \( x \geq 1 \) with the property that \( f'(x) \to 0 \) as \( x \to \infty \).
   a) Show that \( f(n+1) - f(n) \to 0 \) as \( n \to \infty \).
   b) Compute \( \lim_{n \to \infty} \frac{\sqrt{n+1} - \sqrt{n}}{n} \).

4. For \( x \) in any finite interval \( |x| \leq c \) prove that \( \lim_{N \to \infty} \sum_{k=0}^{N} \frac{x^k}{k!} = e^x \) by showing that the remainder in the Taylor series goes to zero.

5. [Error in Interpolation] Let \( f : [a, b] \to \mathbb{R} \) be a smooth function.
   a) Let \( g(x) \) be the straight line with the property that \( g(a) = f(a) \) and \( g(b) = f(b) \).
   For any point \( c \in [a, b] \) obtain an estimate for the error: \( f(c) - g(c) \).
   Remark: Your estimate will involve \( f''(z) \) for some point \( z \in [a, b] \). The estimate is related to the procedure used to find the error in a Taylor polynomial.
HINT: Define the constant $M$ by $f(c) = g(c) + M(c - a)(c - b)$. Then consider the function

$$\phi(x) := f(x) - g(x) - M(x - a)(x - b).$$

b) Let $a = x_0 < x_2 < \cdots < x_k = b$ and let $g(x)$ be the polynomial of degree $k$ that agrees with $f$ at these $k + 1$ points, so $g(x_j) = f(x_j)$, $j = 0, 1, \ldots, k$. Obtain an estimate for the error, $f(c) - g(c)$, for any $c \in [a, b]$. 

6. Let $f : [0, 1] \to \mathbb{R}$ be a continuous function.
   a) If $f(x) \geq 0$ and $\int_0^1 f(x) \, dx = 0$, prove that $f(x) = 0$ for all $x \in [0, 1]$.
   b) If $\int_0^1 f(x) \, dx = 0$, prove that $f(c) = 0$ for some $c \in (0, 1)$. Even more, prove that $f(x)$ changes sign somewhere in this interval.
   c) If $f : [0, 1] \to \mathbb{R}$ is a continuous function with the property that $\int_0^1 f(x)g(x) \, dx = 0$ for all continuous functions $g$ prove that $f(x) = 0$ for all $x \in [0, 1]$.
   d) If $f : [0, 1] \to \mathbb{R}$ is a continuous function with the property that $\int_0^1 f(x)g(x) \, dx = 0$ for all $C^1$ functions $g$ that satisfy $g(0) = g(1) = 0$, must it be true that $f(x) = 0$ for all $x \in [0, 1]$? Proof or counterexample.

7. Let $f(t)$ be a continuous function for $0 \leq t < \infty$. If $\lim_{t \to \infty} f(t) = c$, show that

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T f(t) \, dt = c.$$ 

8. Let $f(x)$ be a continuous function for $0 \leq x \leq 1$. Evaluate $\lim_{n \to \infty} n \int_0^1 f(x)x^n \, dx$. (Justify your assertions.)

9. a) If $V = \{x, y, z\} \in \mathbb{R}^3$ and $p \geq 1$, define $\|V\|_p := \|x\|^p + \|y\|^p + \|z\|^p$ 
   b) If $f \in C[0, 2]$ and $p \geq 1$, define

$$\|f\|_p := \left[ \int_0^2 |f(x)|^p \, dx \right]^{1/p}.$$ 

Show that $\lim_{p \to \infty} \|f\|_p = \max_{0 \leq x \leq 2} |f(x)|$.

10. Compute $\lim_{n \to \infty} \int_0^1 |\sin(nx)| \, dx$. 

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11. Let \( p(x) \) be a real polynomial of degree \( n \). The following uses the inner product 
\[ \langle f, g \rangle := \int_{0}^{1} f(x)g(x) \, dx. \]

a) If \( p \) is orthogonal to the constants, show that \( p \) has at least one real zero in the interval \( \{0 < x < 1\} \).

b) If \( p \) is orthogonal to all polynomials of degree at most one, show that \( p \) has at least two distinct real zeros in the interval \( \{0 < x < 1\} \).

c) If \( p \) is orthogonal to all polynomials of degree at most \( n - 1 \), show that \( p \) has exactly \( n \) distinct real zeros in the interval \( \{0 < x < 1\} \).

**Bonus Problems**

*These are more challenging. If you do any of these, please give your solutions directly to me by Thursday, Nov. 30.*

**Bonus Problem 1** Let \( f : [0, 1] \to \mathbb{R} \) be a continuous function.

a) Show that \( \lim_{\lambda \to \infty} \int_{0}^{1} f(x) \sin(\lambda x) \, dx = 0. \)

b) (generalization) If \( \phi : \mathbb{R} \to \mathbb{R} \) is continuous with period \( P \), show that

\[ \lim_{\lambda \to \infty} \int_{0}^{1} f(x)\phi(\lambda x) \, dx = \Phi \int_{0}^{1} f(x) \, dx, \]

where \( \Phi := \frac{1}{P} \int_{0}^{P} \phi(t) \, dt \) is the average of \( \phi \) over one period.

**Bonus Problem 2** Let \( C \) be the ring of continuous functions on the interval \( 0 \leq x \leq 1 \).

a) If \( 0 \leq c \leq 1 \), show that the subset \( \{ f \in C \mid f(c) = 0 \} \) is a maximal ideal.

b) Show that every maximal ideal in \( C \) has this form.

**Bonus Problem 3** Let \( a_0, a_1, \ldots \) be any sequence of real numbers. Show there is a smooth function \( f(x) \) with the property that \( a_n \) is its \( n \)th Taylor coefficient: \( a_n = \frac{1}{n!} f^{(n)}(x) \big|_{x=0} \).