Directions: This exam has two parts, Part A has 10 True-False problems (30 points, 3 points each). Part B has 5 traditional problems (70 points, 14 points each). Closed book, no calculators or computers— but you may use one 3" x 5" card with notes on both sides.

Part A: True/False (answer only, no reasons). 10 problems, 3 points each.
Circle T or F in each problem.

1. A bounded sequence \( \{a_n\} \) of real numbers always has a convergent subsequence.

2. A series \( \sum_{n=1}^{\infty} a_n \) of complex numbers converges if and only if the corresponding sequence of partial sums is bounded.

3. A closed and bounded subset of a complete metric space must be compact.

4. If \( A \) and \( B \) are compact subsets of a metric space, then \( A \cup B \) is also compact.

5. If \( M \) is any metric space and \( f : M \to \mathbb{R} \) is any continuous real-valued function, then the function \( g : M \to \mathbb{R} \) defined by \( g(x) := (f(x))^2 \) is always continuous.

6. If \( f : X \to Y \) is a continuous map between metric spaces, and \( f(X) \) is compact, then \( X \) is compact.

7. A compact subset of a metric space is always complete.

8. Let \( \{x_n\} \) be a sequence of points in a metric space. If two subsequences of this sequence converge, then they must converge to the same number.

9. If \( f : [0, 1] \to \mathbb{R} \) is a continuous function and \( \int_0^1 f(x) \, dx = 0 \), then \( f(x) \) is positive somewhere and negative somewhere in this interval (unless it is identically zero).

10. \( f(x) := \sum_{n=1}^{\infty} \frac{\sin(3^n \pi x)}{2^n} \) is a continuous function on \( \mathbb{R} \).
Part B: Traditional Problems (5 problems, 14 points each)

B–1. Let \( f : [-2, 2] \) be a smooth function with the property that
\[
    f(-1) = 1, \quad f(0) = 0, \quad f(1) = 2.
\]
Show that at some point \( c \in (-1, 1) \) we have \( f''(c) > 0 \). In fact, find an explicit constant \( m > 0 \) so that \( f''(c) \geq m \).

B–2. Let \( A(t) \) and \( B(t) \) be \( n \times n \) matrices that are differentiable for \( t \in [a, b] \) and let \( t_0 \in (a, b) \).
Directly from the definition of the derivative, show that the product \( M(t) := A(t)B(t) \) is differentiable at \( t = t_0 \) and obtain the usual formula for \( M'(t_0) \).

B–3. Let \( w(x) \) be a smooth function that satisfies \( w'' - c(x)w = 0 \), where \( c(x) > 0 \) is a given function.
   a) Show that \( w \) cannot have a local positive maximum (that is, a maximum at an interior point where the function is positive). Also show that \( w \) cannot have a local negative minimum.
   b) [Uniqueness] If you also know that \( w(0) = a \) and \( w(1) = b \), prove that there is at most one solution \( w(x) \in C^2([0, 1]) \) with these properties.

B–4. Let \( f(x) \) and \( K(x, y) \) be a given continuous real valued functions for \( x, y \in [0, 2] \), and, say \( |K(x, y)| \leq M \). Show that if \( 0 < a \leq 2 \) is sufficiently small, the integral equation
\[
    u(x) = f(x) + \int_0^x K(x, y) u(y) \, dy
\]
has a unique continuous solution \( u(x) \) for \( x \in [0, a] \).

B–5. Let \( \varphi_n(t) \) be a sequence of smooth real-valued functions with the properties
\[
    (a) \ \varphi_n(t) \geq 0, \quad (b) \ \varphi_n(t) = 0 \ \text{for} \ |t| \geq 1/n, \quad (c) \ \int_{-\infty}^{\infty} \varphi_n(t) \, dt = 1.
\]
Note: because of (b), this integral is only over \(-1/n \leq t \leq 1/n\).
Assume \( f(x) \) is uniformly continuous for all \( x \in \mathbb{R} \) and define
\[
    f_n(x) := \int_{-\infty}^{\infty} f(x-t)\varphi_n(t) \, dt.
\]
Show that \( f_n(x) \) converges uniformly to \( f(x) \) for all \( x \in \mathbb{R} \). [Suggestion: Use \( f(x) = f(x) \left( \int_{-\infty}^{\infty} \varphi_n(t) \, dt \right) = \int_{-\infty}^{\infty} f(x)\varphi_n(t) \, dt \). Also, note explicitly where you use the uniform continuity of \( f \).]
Remark: One can show that the approximations \( f_n \) are also smooth. Thus, this proves that you can approximate a continuous function uniformly on any compact set by a smooth function.