Math 508 | Exam 1 | Jerry L. Kazdan
October 14, 2010 | | 9:00 – 10:20

Exam 1

DIRECTIONS This exam has three parts, Part A asks for 4 examples (20 points, 5 points each). Part B has 4 shorter problems (36 points, 9 points each. Part C has 3 traditional problems (45 points, 15 points each). Total is 101 points.
Closed book, no calculators or computers– but you may use one $3'' \times 5''$ card with notes on both sides.

Part A: Examples (4 problems, 5 points each so 20 points). Give an example having the specified property.

1. A subset of the real line $\mathbb{R}$ that contains all but three of its limit points.

2. A sequence of real numbers x_n with the property that $|x_{n+1} - x_n| \to 0$ but the sequence does *not* converge.

3. A collection of closed intervals $A_k \subset \mathbb{R}$, $k = 1, 2 \ldots$, whose union is not closed.

4. A collection of bounded nested intervals $J_1 \supseteq J_2 \supseteq J_3 \supseteq \cdots$ whose intersection is empty.

Part B: Short Problems (4 problems, 9 points each so 36 points)

B–1. Let $a_k = \sqrt{k+1} - \sqrt{k}, \quad k = 1, 2, \ldots$. Determine if this sequence converges.

B–2. Show that $1 + \dfrac{1}{2!} + \dfrac{1}{3!} + \dfrac{1}{4!} + \ldots + \dfrac{1}{k!} + \ldots < 2$

B–3. Given any two rational numbers $p < q$, prove there is an *irrational* number c between them.

B–4. Given any $c > 1$, show that $\dfrac{c^n}{n!} \to 0$.

Part C: Traditional Problems (3 problems, 15 points each so 45 points)

C–1. Let $x_k \geq 0$ be a sequence of real numbers that converges to A. Show that $A \geq 0$.

C–2. Given a sequence $\{a_k\}$ of real numbers, let $S_n = \dfrac{a_1 + \cdots + a_n}{n}$ be the sequence of averages (*arithmetic mean*). If a_k converges to 0, show that the averages S_n also converge to 0.

C–3. Let $\{a_n\} \in \mathbb{C}$ be a *contracting* sequence, that is there is a $0 < c < 1$ so that

$$|a_{n+1} - a_n| \le c|a_n - a_{n-1}|, \quad n = 1, 2, 3, \ldots.$$

a) Show that $\quad |a_{n+1} - a_n| \le c^n |a_1 - a_0|$.

b) If $n > k$, show that $\quad |a_n - a_k| \le \dfrac{c^k}{1-c}|a_1 - a_0|$.

c) Show that the sequence a_n converges.