Math 508 | **Exam 2** | Jerry L. Kazdan
December 9, 2010 | | 9:00 – 10:20

DIRECTIONS This exam has three parts, Part A asks for 3 examples (5 points each, so 15 points). Part B has 4 shorter problems (8 points each so 32 points). Part C has 4 traditional problems (15 points each so 60 points). Total is 107 points.
Closed book, no calculators or computers– but you may use one $3'' \times 5''$ card with notes on both sides.

Part A: Examples (3 examples, 5 points each so 15 points). Give an example having the specified property.

1. A function $f \in C^1([-1,1])$ but is not in $C^2([-1,1])$.

2. A bounded sequence a_k in a complete metric space $\mathcal{M}$ where a_k has no convergent subsequence.

3. A sequence of continuous functions $f_n(x) \in C([0,1])$ that converges pointwise to zero but $\int_0^1 f_n(x)\,dx \ge 1$. [A clear sketch is adequate.]

Part B: Short Problems (4 problems, 8 points each so 32 points)

B–1. Let $f(x)$ be a smooth function with the properties: $f(-1) = 1$, $f(0) = 0$, and $f(1) = 1$. Show that $f''(c) = 2$ at some $c \in (-1,\, 1)$. [Suggestion: Consider $g(x) := f(x) - x^2$.]

B–2. Let $\displaystyle\int_0^{2x} f(t)\,dt = e^{\cos(3x+1)} + A$. Find $f \in C(\mathbb{R})$ and the constant A.

B–3. Let $f \in C([1,3])$. Compute $\displaystyle\lim_{n\to\infty} \int_1^3 f(x)e^{-nx}\,dx$. [Justify your assertions.]

B–4. Show that $\displaystyle f(x) := \sum_1^\infty \frac{\sin(3nx^2)}{n^2}$ is continuous for $0 \le x \le \pi$.

Part C: Traditional Problems (4 problems, 15 points each so 60 points)

C–1. Let $A(t)$ and $B(t)$ be $n \times n$ matrices that are differentiable for $t \in [a,\, b]$ and let $t_0 \in (a,\, b)$. Directly from the *definition* of the derivative, show that the product $M(t) := A(t)B(t)$ is differentiable at $t = t_0$ and obtain the usual formula for $M'(t_0)$.

C–2. Let K be a compact set in a complete metric space $\mathcal{M}$ with metric $d(x,y)$. If $p \in \mathcal{M}$ is a point *not* in K, let $c = \inf_{x\in K} d(p,x)$. Show there is a point $q \in K$ such that $d(p,q) = c$.

C–3. Let $f \in C^1([0,2])$. Given any $\epsilon > 0$ show there is a polynomial $p(x)$ such that

$$\max_{x\in[0,2]} |f(x) - p(x)| + \max_{x\in[0,2]} |f'(x) - p'(x)| < \epsilon$$

That is, $\|f - p\|_{C^1([0,2])} < \epsilon$.

C–4. Let $f(x)$ and $h(x,y)$ be continuous functions for $x, y \in [0,2]$. Show that if the constant $\lambda > 0$ is sufficiently small, the equation

$$u(x) = f(x) + \lambda \int_0^2 h(x,y)u(y)\,dy.$$

has a unique solution $u(x) \in C([0,2])$.

Extra Problems The following are some problems that I almost put on the exam — but then it would have been much too long.

Ex–1. Let $0 < a_n \in \mathbb{R}$ be a sequence with the property that $\dfrac{a_{n+1}}{a_n} \le c$, $n = 1, 2, \ldots$ for some $0 < c < 1$. Show that $a_n \to 0$.

Ex–2. Show that a compact set in a metric space is bounded.

Ex–3. Let $\mathbb{R}^2$ be the points $V = (x,y)$ with the usual Euclidean norm $\|V\| = \sqrt{x^2 + y^2}$. Using that $\mathbb{R}$ is complete with norm $|x|$, prove directly that $\mathbb{R}^2$ is complete.

Ex–4. If $\displaystyle\sum_0^\infty a_n z^n$ converges at $z = R$ and if $0 < r < R$, prove that it converges uniformly in the disk $\{z \in \mathbb{C} : |z| \le r\}$.

Ex–5. Let $\varphi_n(t)$ be a sequence of smooth real-valued functions with the properties

$$(a)\ \varphi_n(t) \ge 0, \qquad (b)\ \varphi_n(t) = 0 \text{ for } |t| \ge 1/n, \qquad (c)\ \int_{-\infty}^{\infty} \varphi_n(t)\,dt = 1.$$

Note: because of (b), this integral is only over $-1/n \le t \le 1/n$.

Assume $f(x)$ is uniformly continuous for all $x \in \mathbb{R}$ and define

$$f_n(x) := \int_{-\infty}^{\infty} f(x-t)\varphi_n(t)\,dt.$$

Show that $f_n(x)$ converges uniformly to $f(x)$ for all $x \in \mathbb{R}$. Note *explicitly* where you use the uniform continuity of f.

[Suggestion: Use $f(x) = f(x)\left(\int_{-\infty}^{\infty} \varphi_n(t)\,dt\right) = \int_{-\infty}^{\infty} f(x)\varphi_n(t)\,dt$].