DIRECTIONS This exam has three parts, Part A asks for 3 examples (5 points each, so 15 points). Part B has 4 shorter problems (8 points each so 32 points). Part C has 4 traditional problems (15 points each so 60 points). Total is 107 points.

Closed book, no calculators or computers— but you may use one 3″ × 5″ card with notes on both sides.

Part A: Examples (3 examples, 5 points each so 15 points). Give an example having the specified property.

1. A function \( f \in C^1([-1,1]) \) but is not in \( C^2([-1,1]) \).

   SOLUTION: \( f(x) := x|x| \)

2. A bounded sequence \( a_k \) in a complete metric space \( M \) where \( a_k \) has no convergent subsequence.

   SOLUTION: In \( \ell_2 \) the unit vectors \( e_1 := (1,0,0,\ldots), e_2 := (0,1,0,\ldots), \ldots \).

   Another example: In \( C([0,1]) \) the functions \( f_k(x) := x^k, k = 0, 1, 2, \ldots \) [since every subsequence converges pointwise to the discontinuous function \( f(x) := 0, x \in [0,1) \) but \( f(1) = 1 \)].

   Another example: In \( L_2((-\pi, \pi)) \), the functions \( \sin kx/\sqrt\pi, k = 1, 2, \ldots \) [since they are orthonormal].

3. A sequence of continuous functions \( f_n(x) \in C([0,1]) \) that converges pointwise to zero but \( \int_0^1 f_n(x) \, dx \geq 1 \). [A clear sketch is adequate.]

   SOLUTION: Let \( f_n(x) \in C([0,1]) \) be the “tent” function whose graph is straight lines from \((0,0)\) to \((1/n, n)\) to \((2/n, 0)\) to \((1,0)\).

Part B: Short Problems (4 problems, 8 points each so 32 points)

B–1. Let \( f(x) \) be a smooth function with the properties: \( f(-1) = 1, f(0) = 0, \) and \( f(1) = 1 \). Show that \( f''(c) = 2 \) at some \( c \in (-1, 1) \). [Suggestion: Consider \( g(x) := f(x) - x^2 \).]

   SOLUTION: Let \( g(x) := f(x) - x^2 \). Then \( g(-1) = g(0) = g(1) = 0 \). By the mean value theorem there is at least one point \( c_1 \in (-1, 0) \) where \( g'(c_1) = 0 \) and \( c_2 \in (0, 1) \) where \( g'(c_2) = 0 \). Applying the mean value theorem a third time, this time to \( g'(x) \) there is a point \( c \in (c_1, c_2) \) where \( g''(c) = 0 \). But \( g''(c) = f''(c) - 2 \).

B–2. Let \( \int_0^{2x} f(t) \, dt = e^{\cos(3x+1)} + A \). Find \( f \in C(\mathbb{R}) \) and the constant \( A \).

   SOLUTION: Let \( x = 0 \) to see that \( A = -e^{\cos 1} \). Take the derivative of both sides with respect to \( x \) to find that \( 2f(2x) = e^{\cos(3x+1)}(-3\sin(3x+1)) \) so

\[
f(x) = -\frac{3}{2} \sin(3x/2 + 1)e^{\cos((3x/2)+1)}.
\]
Alternate: First make the substitution \( w := 2x \) in the original equation:

\[
\int_0^w f(t) \, dt = e^{\cos(3w/2+1)} + A.
\]

Now let \( w = 0 \) to find \( A \) and take the derivative of both sides with respect to \( w \) to find \( f(w) \).

B–3. Let \( f \in C([1,3]) \). Compute \( \lim_{n \to \infty} \int_1^3 f(x)e^{-nx} \, dx \). [Justify your assertions.]

Solution: Since \( f \in C([1,3]) \), it is bounded, so say \(|f(x)| \leq M \) in \([1,3]\). Then

\[
\left| \int_1^3 f(x)e^{-nx} \, dx \right| \leq M \int_1^3 e^{-nx} \, dx \leq 2Me^{-n} \to 0.
\]

Alternate: Observe that the sequence \( \lim_{n \to \infty} f(x)e^{-nx} = 0 \) uniformly on the bounded interval \([1,3]\) so we can interchange limit and integral.

B–4. Show that \( f(x) := \sum_{1}^{\infty} \frac{\sin(3nx^2)}{n^2} \) is continuous for \( 0 \leq x \leq \pi \).

Solution: Since \(|\sin(3nx^2)| \leq 1 \) and \( \sum \frac{1}{n^2} \) converges, the series converges uniformly by the Weierstrass M-Test. Now use that the uniform limit of continuous functions is continuous.

Part C: Traditional Problems (4 problems, 15 points each so 60 points)

C–1. Let \( A(t) \) and \( B(t) \) be \( n \times n \) matrices that are differentiable for \( t \in [a, b] \) and let \( t_0 \in (a, b) \).

Directly from the definition of the derivative, show that the product \( M(t) := A(t)B(t) \) is differentiable at \( t = t_0 \) and obtain the usual formula for \( M'(t_0) \).

Solution: From the definition of the derivative, we need to examine

\[
\lim_{h \to 0} \frac{M(t_0 + h) - M(t_0)}{h} = \lim_{h \to 0} \frac{A(t_0 + h)B(t_0 + h) - A(t_0)B(t_0)}{h}
\]

But

\[
\frac{A(t_0 + h)B(t_0 + h) - A(t_0)B(t_0)}{h} = \frac{[A(t_0 + h) - A(t_0)]B(t_0 + h)}{h} + \frac{A(t_0)[B(t_0 + h) - B(t_0)]}{h}
\]

\[
\to A'(t_0)B(t_0) + A(t_0)B'(t_0)
\]

Therefore \( M'(t_0) \) exist and equals \( A'(t_0)B(t_0) + A(t_0)B'(t_0) \).

C–2. Let \( K \) be a compact set in a complete metric space \( M \) with metric \( d(x,y) \). If \( p \in M \) is a point not in \( K \), let \( c = \inf_{x \in K} d(p,x) \). Show there is a point \( q \in K \) such that \( d(p,q) = c \).

Thus, \( q \) is a point in \( K \) that is closest to \( p \),
**Solution:** We use that for a compact set in a metric space, every sequence has a convergent subsequence. From the definition of $c$, there is a sequence $x_n \in K$ such that $d(p, x_n) \to c$. The $x_n$ has a subsequence $x_{n_j}$ that converges to some $q \in K$. Thus

$$c = \lim d(p, x_{n_j}) = d(p, q).$$

Here we used that $d(x, y)$ is a continuous function of $y$ (and similarly, $x$). This follows from $|d(x, y) - d(x, z)| \leq d(y, z)$.

The example where $K \subset \mathbb{R}^2$ is the annulus: $(x, y) \in \mathbb{R}^2 : 1 \leq x^2 + y^2 \leq 4$ and $p = (0, 0)$ is the origin shows that there may be many closest points $q \in K$ so using a convergent subsequence is essential.

**C–3.** Let $f \in C^1([0, 2])$. Given any $\epsilon > 0$ show there is a polynomial $p(x)$ such that

$$\max_{x \in [0, 2]} |f(x) - p(x)| + \max_{x \in [0, 2]} |f'(x) - p'(x)| < \epsilon \quad (1)$$

That is, $\|f - p\|_{C^1([0, 2])} < \epsilon$.

**Solution:** Idea: first approximate $f'$ by a polynomial $q(x)$. Then integrate to approximate $f(x)$. [Approximating $f$ first can’t work since although two functions can be fairly close in the uniform norm, their derivatives may be far apart. Example: $f(x) = \frac{\sin 1000x}{10}$ and $g(x) := 0$].

In greater detail, given any $\epsilon > 0$, by the Weierstrass Approximation Theorem there is a polynomial $q(x)$ with $|f'(x) - q(x)| < \epsilon/3$ for all $x \in [0, 2]$. Let $p(x) := f(0) + \int_0^x q(t) \, dt$ so $p' = q$. Since $f(x) = f(0) + \int_0^x f'(t) \, dt$, then

$$|f(x) - p(x)| = \left| \int_0^x [f'(t) - p'(t)] \, dt \right| \leq \int_0^x |f'(t) - q(t)| \, dt \leq 2\epsilon/3.$$

Thus (1) is satisfied.

**C–4.** Let $f(x)$ and $h(x, y)$ be continuous functions for $x, y \in [0, 2]$. Show that if the constant $\lambda > 0$ is sufficiently small, the equation

$$u(x) = f(x) + \lambda \int_0^2 h(x, y) u(y) \, dy.$$

has a unique solution $u(x) \in C([0, 2])$.

**Solution:** Let $\mathcal{M}$ be $C([0, 2])$ with the uniform norm. This is complete since the uniform limit of continuous functions is continuous.

Define the map

$$T \varphi(x) := f(x) + \lambda \int_0^2 h(x, y) \varphi(y) \, dy.$$
Since \( h(x, y) \) is assumed continuous for \( x, y \in [0, 2] \) we see that \( T : \mathcal{M} \to \mathcal{M} \). Thus we need only show that for small \( \lambda \) the map \( T \) is contracting. Because \( h(x, y) \) is continuous and \([0, 2] \times [0, 2]\) is Compact, \( h(x, y) \) is bounded, say \( |h(x, y)| \leq M \) for all \( x, y \in [0, 2] \). Then

\[
|T\varphi(x) - T\psi(x)| \leq \lambda \int_0^2 |h(x, y)||\varphi(x) - \psi(x)| \, dx \\
\leq 2\lambda M \|\varphi - \psi\|_\infty.
\]

Picking \( \lambda < 1/(2M) \) it is clear the contracting condition is satisfied.
**Extra Problems** The following are some problems that I almost put on the exam — but then it would have been much too long.

Ex–1. Let $0 < a_n \in \mathbb{R}$ be a sequence with the property that $\frac{a_{n+1}}{a_n} \leq c$, $n = 1, 2, \ldots$ for some $0 < c < 1$. Show that $a_n \to 0$.

Ex–2. Show that a compact set in a metric space is bounded.

Ex–3. Let $\mathbb{R}^2$ be the points $V = (x, y)$ with the usual Euclidean norm $\|V\| = \sqrt{x^2 + y^2}$. Using that $\mathbb{R}$ is complete with norm $|x|$, prove directly that $\mathbb{R}^2$ is complete.

Ex–4. If $\sum_{0}^{\infty} a_n z^n$ converges at $z = 1$ and if $0 < r < 1$, prove that it converges uniformly in the disk $\{z \in \mathbb{C} : |z| \leq r\}$.

Ex–5. Let $\varphi_n(t)$ be a sequence of smooth real-valued functions with the properties

\begin{align*}
(a) \varphi_n(t) \geq 0, & \quad (b) \varphi_n(t) = 0 \text{ for } |t| \geq 1/n, \quad (c) \int_{-\infty}^{\infty} \varphi_n(t) \, dt = 1.
\end{align*}

Note: because of (b), this integral is only over $-1/n \leq t \leq 1/n$.

Assume $f(x)$ is uniformly continuous for all $x \in \mathbb{R}$ and define

$$f_n(x) := \int_{-\infty}^{\infty} f(x - t) \varphi_n(t) \, dt.$$

Show that $f_n(x)$ converges uniformly to $f(x)$ for all $x \in \mathbb{R}$. [Suggestion: Use $f(x) = f(x) \left( \int_{-\infty}^{\infty} \varphi_n(t) \, dt \right) = \int_{-\infty}^{\infty} f(x) \varphi_n(t) \, dt$. Also, note explicitly where you use the uniform continuity of $f$.]