Math 508 **Exam 2** Jerry L. Kazdan
December 9, 2010 9:00 – 10:20

DIRECTIONS This exam has three parts, Part A asks for 3 examples (5 points each, so 15 points). Part B has 4 shorter problems (8 points each so 32 points). Part C has 4 traditional problems (15 points each so 60 points). Total is 107 points.
Closed book, no calculators or computers– but you may use one $3'' \times 5''$ card with notes on both sides.

Part A: Examples (3 examples, 5 points each so 15 points). Give an example having the specified property.

1. A function $f \in C^1([-1,1])$ but is not in $C^2([-1,1])$.

   SOLUTION: $f(x) := x|x|$

2. A bounded sequence a_k in a complete metric space $\mathcal{M}$ where a_k has no convergent subsequence.

   SOLUTION: In ℓ_2 the unit vectors $e_1 := (1,0,0,\ldots)$, $e_2 := (0,1,0,\ldots)$, $\ldots$.
   Another example: In $C([0,1])$ the functions $f_k(x) := x^k$, $k = 0, 1, 2, \ldots$ [since every subsequence converges pointwise to the discontinuous function $f(x) := 0$, $x \in [0.1)$ but $f(1) = 1$.
   Another example: In $L_2((-\pi, \pi))$, the functions $\frac{\sin kx}{\sqrt{\pi}}$, $k = 1, 2, \ldots$ [since they are orthonormal].

3. A sequence of continuous functions $f_n(x) \in C([0,1])$ that converges pointwise to zero but $\int_0^1 f_n(x)\,dx \ge 1$. [A clear sketch is adequate.]

   SOLUTION: Let $f_n(x) \in C([0,1])$ be the "tent" function whose graph is straight lines from $(0,0)$ to $(1/n,\, n)$ to $(2/n,\, 0)$ to $(1,0)$.

Part B: Short Problems (4 problems, 8 points each so 32 points)

B–1. Let $f(x)$ be a smooth function with the properties: $f(-1) = 1$, $f(0) = 0$, and $f(1) = 1$. Show that $f''(c) = 2$ at some $c \in (-1,\, 1)$. [Suggestion: Consider $g(x) := f(x) - x^2$.]

SOLUTION: Let $g(x) := f(x) - x^2$. Then $g(-1) = g(0) = g(1) = 0$. By the mean value theorem there is at least one point point $c_1 \in (-1,\, 0)$ where $g'(c_1) = 0$ and $c_2 \in (0,\, 1)$ where $g'(c_2) = 0$. Applying the mean value theorem a third time, this time to $g'(x)$ there is a point $c \in (c_1,\, c_2)$ where $g''(c) = 0$. But $g''(c) = f''(c) - 2$.

B–2. Let $\displaystyle\int_0^{2x} f(t)\,dt = e^{\cos(3x+1)} + A$. Find $f \in C(\mathbb{R})$ and the constant A.

SOLUTION: Let $x = 0$ to see that $A = -e^{\cos 1}$. Take the derivative of both sides with respect to x to find that $2f(2x) = e^{\cos(3x+1)}(-3\sin(3x+1))$ so

$$f(x) = -\frac{3}{2}\sin[(3x/2)+1]e^{\cos[(3x/2)+1]}.$$

ALTERNATE: First make the substitution $w := 2x$ in the original equation:

$$\int_0^w f(t)\,dt = e^{\cos(3w/2+1)} + A.$$

Now let $w = 0$ to find A and take the derivative of both sides with respect to w to find $f(w)$.

B–3. Let $f \in C([1,3])$. Compute $\lim_{n\to\infty} \int_1^3 f(x)e^{-nx}\,dx$. [Justify your assertions.]

SOLUTION: Since $f \in C([1,3])$, it is bounded, so say $|f(x)| \le M$ in $[1,3]$. Then

$$\left|\int_1^3 f(x)e^{-nx}\,dx\right| \le M \int_1^3 e^{-nx}\,dx \le 2Me^{-n} \to 0.$$

ALTERNATE: Observe that the sequence $\lim_{n\to\infty} f(x)e^{-nx} = 0$ *uniformly* on the bounded interval $[1,3]$ so we can interchange limit and integral.

B–4. Show that $f(x) := \sum_1^\infty \frac{\sin(3nx^2)}{n^2}$ is continuous for $0 \le x \le \pi$.

SOLUTION: Since $|\sin(3nx^2)| \le 1$ and $\sum 1/(n^2)$ converges, the series converges uniformly by the Weierstrass M-Test. Now use that the uniform limit of continuous functions is continuous.

Part C: Traditional Problems (4 problems, 15 points each so 60 points)

C–1. Let $A(t)$ and $B(t)$ be $n \times n$ matrices that are differentiable for $t \in [a,\, b]$ and let $t_0 \in (a,\, b)$. Directly from the *definition* of the derivative, show that the product $M(t) := A(t)B(t)$ is differentiable at $t = t_0$ and obtain the usual formula for $M'(t_0)$.

SOLUTION: From the definition of the derivative, we need to examine

$$\lim_{h\to 0} \frac{M(t_0+h) - M(t_0)}{h} = \lim_{h\to 0} \frac{A(t_0+h)B(t_0+h) - A(t_0)B(t_0)}{h}$$

But

$$\begin{aligned}\frac{A(t_0+h)B(t_0+h) - A(t_0)B(t_0)}{h} &= \frac{[A(t_0+h) - A(t_0)]B(t_0+h)}{h} + \frac{A(t_0)[B(t_0+h) - B(t_0)]}{h} \\ &\to A'(t_0)B(t_0) + A(t_0)B'(t_0)\end{aligned}$$

Therefore $M'(t_0)$ exist and equals $A'(t_0)B(t_0) + A(t_0)B'(t_0)$.

C–2. Let K be a compact set in a complete metric space $\mathcal{M}$ with metric $d(x,y)$. If $p \in \mathcal{M}$ is a point *not* in K, let $c = \inf_{x\in K} d(p,x)$. Show there is a point $q \in K$ such that $d(p,q) = c$. Thus, q is a point in K that is closest to p, SOLUTION: We use that for a compact set in a metric space, every sequence has a convergent subsequence. From the definition of c, there is a sequence $x_n \in K$ such that $d(p, x_n) \to c$. The x_n has a subsequence x_{n_j} that converges to some $q \in K$. Thus

$$c = \lim d(p, x_{n_j}) = d(p, q).$$

Here we used that $d(x, y)$ is a continuous function of y (and similarly, x). This follows from $|d(x, y) - d(x, z)| \leq d(y, z)$.

The example where $K \subset \mathbb{R}^2$ is the annulus: $\{(x, y) \in \mathbb{R}^2 : 1 \leq x^2 + y^2 \leq 4\}$ and $p = (0, 0)$ is the origin shows that there may be many closest points $q \in K$ so using a convergent subsequence is essential.

C–3. Let $f \in C^1([0, 2])$. Given any $\epsilon > 0$ show there is a polynomial $p(x)$ such that

$$\max_{x \in [0,2]} |f(x) - p(x)| + \max_{x \in [0,2]} |f'(x) - p'(x)| < \epsilon \tag{1}$$

That is, $\|f - p\|_{C^1([0,2])} < \epsilon$.

SOLUTION: Idea: first approximate f' by a polynomial $q(x)$. Then integrate to appromimate $f(x)$. [Approximating f first *can't work* since although two functions can be fairly close in the uniform norm, their derivatives may be far apart. Example: $f(x) = \dfrac{\sin 1000x}{10}$ and $g(x) := 0$].

In greater detail, given any $\epsilon > 0$, by the Weierstrass Approximation Theorem there is a polynomial $q(x)$ with $|f'(x) - q(x)| < \epsilon/3$ for all $x \in [0, 2]$. Let $p(x) := f(0) + \int_0^x q(t)\, dt$ so $p' = q$. Since $f(x) = f(0) + \int_0^x f'(t)\, dt$, then

$$\begin{aligned} |f(x) - p(x)| &= \left| \int_0^x [f'(t) - p'(t)]\, dt \right| \\ &\leq \int_0^2 |f'(t) - q(t)|\, dt \leq 2\epsilon/3. \end{aligned}$$

Thus (1) is satisfied.

C–4. Let $f(x)$ and $h(x, y)$ be continuous functions for $x, y \in [0, 2]$. Show that if the constant $\lambda > 0$ is sufficiently small, the equation

$$u(x) = f(x) + \lambda \int_0^2 h(x, y) u(y)\, dy.$$

has a unique solution $u(x) \in C([0, 2])$.

SOLUTION: Let $\mathcal{M}$ be $C([0, 2])$ with the uniform norm. This is complete since the uniform limit of continuous functions is continuous.

Define the map

$$T\varphi(x) := f(x) + \lambda \int_0^2 h(x, y) \varphi(y)\, dy.$$

Since $h(x,y)$ is assumed continuous for $x, y \in [0,2]$ we see that $T : \mathcal{M} \to \mathcal{M}$. Thus we need only show that for small λ the map T is contracting. Because $h(x,y)$ is continuous and $[0,2] \times [0,2]$ is Compact, $h(x,y)$ is bounded, say $|h(x,y| \leq M$ for all $x, y \in [0,2]$. Then

$$\begin{aligned}|T\varphi(x) - T\psi(x)| \leq& \lambda \int_0^2 |h(x,y)[\varphi(x) - \psi(x)]| \, dx \\ \leq& 2\lambda M \|\varphi - \psi\|_\infty.\end{aligned}$$

Picking $\lambda < 1/(2M)$ it is clear the contracting condition is satisfied.

Extra Problems The following are some problems that I almost put on the exam — but then it would have been much too long.

Ex–1. Let $0 < a_n \in \mathbb{R}$ be a sequence with the property that $\dfrac{a_{n+1}}{a_n} \le c$, $n = 1, 2, \ldots$ for some $0 < c < 1$. Show that $a_n \to 0$.

Ex–2. Show that a compact set in a metric space is bounded.

Ex–3. Let $\mathbb{R}^2$ be the points $V = (x, y)$ with the usual Euclidean norm $\|V\| = \sqrt{x^2 + y^2}$. Using that $\mathbb{R}$ is complete with norm $|x|$, prove directly that $\mathbb{R}^2$ is complete.

Ex–4. If $\displaystyle\sum_0^\infty a_n z^n$ converges at $z = 1$ and if $0 < r < 1$, prove that it converges uniformly in the disk $\{z \in \mathbb{C} : |z| \le r\}$.

Ex–5. Let $\varphi_n(t)$ be a sequence of smooth real-valued functions with the properties

$$(a)\ \varphi_n(t) \ge 0, \qquad (b)\ \varphi_n(t) = 0 \ \text{ for } \ |t| \ge 1/n, \qquad (c)\ \int_{-\infty}^{\infty} \varphi_n(t)\, dt = 1.$$

Note: because of (b), this integral is only over $-1/n \le t \le 1/n$.

Assume $f(x)$ is uniformly continuous for all $x \in \mathbb{R}$ and define

$$f_n(x) := \int_{-\infty}^{\infty} f(x - t)\varphi_n(t)\, dt.$$

Show that $f_n(x)$ converges uniformly to $f(x)$ for all $x \in \mathbb{R}$. [SUGGESTION: Use $f(x) = f(x)\left(\int_{-\infty}^{\infty} \varphi_n(t)\, dt\right) = \int_{-\infty}^{\infty} f(x)\varphi_n(t)\, dt$. Also, note *explicitly* where you use the uniform continuity of f].