Completeness of $\ell_1$

Let $\ell_1$ be the vector space of infinite sequences of real numbers $X = (x_1, x_2, \ldots)$ with finite norm $\|X\| := \sum_{j=1}^{\infty} |x_j|$. Here we show this space is complete. The proof is a bit fussy.

**STEP 1: FIND A CANDIDATE FOR THE LIMIT.** Let $X_n = (x_1^{(n)}, x_2^{(n)}, \ldots)$ be a Cauchy sequence. Because

$$|x_1^{(n)} - x_2^{(n)}| \leq \|X_n - X_{\ell}\|,$$

the first coordinates $x_1^{(n)}$ are a Cauchy sequence of real numbers and hence converge to some real number $z_1$. Similarly, the other coordinates converge: $\lim_{n \to \infty} x_j^{(n)} \to z_j$ for some real numbers $z_j$. We thus suspect that $X_n \to Z := (z_1, z_2, \ldots)$.

**STEP 2: SHOW THAT THIS CANDIDATE Z IS IN $\ell_1$.** We show that $\|Z\| = \sum |z_j| = \lim_{n \to \infty} \sum_{j=1}^{J} |z_j| < \infty$.

Now

$$\sum_{j=1}^{J} |z_j| = \sum_{j=1}^{J} \lim_{n \to \infty} |x_j^{(n)}| = \lim_{n \to \infty} \sum_{j=1}^{J} |x_j^{(n)}|.$$}

Note that in the second equality there are no difficulties interchanging a limit with the sum of a finite number of real numbers. Since Cauchy sequences are bounded, there is an $M$ such that $|X_n| < M$ for all $n$. Thus, for any $J$

$$\sum_{j=1}^{J} |x_j^{(n)}| \leq \sum_{j=1}^{\infty} |x_j^{(n)}| = \|X_n\| < M.$$}

Letting $n \to \infty$ we find that $\sum_{j=1}^{J} |z_j| \leq \|X_n\| < M$. Because $J$ is arbitrary, we conclude that $\|Z\| \leq M$ and hence $Z \in \ell_1$.

**STEP 3: PROVE THE CONVERGENCE.** Let $X_n$ be a Cauchy sequence. Given $\varepsilon > 0$ pick $N$ so that if $n$ and $\ell > N$ then $\|X_n - X_{\ell}\| < \varepsilon$. Consequently, for any fixed $J$ and $n, \ell > N$, we find

$$\sum_{j=1}^{J} |x_j^{(n)} - x_j^{(\ell)}| \leq \sum_{j=1}^{\infty} |x_j^{(n)} - x_j^{(\ell)}| = \|X_n - X_{\ell}\| < \varepsilon.$$}

With $n > N$ and $J$ fixed, let $\ell \to \infty$ to find

$$\sum_{j=1}^{J} |x_j^{(n)} - z_j| = \lim_{\ell \to \infty} \sum_{j=1}^{J} |x_j^{(n)} - x_j^{(\ell)}| \leq \varepsilon.$$}

But now this is true for all $J$ so $\|X_n - Z\| \leq \varepsilon$, as desired.

**Exercise** Let $\ell_2$ be the vector space of infinite sequences of real numbers $X = (x_1, x_2, \ldots)$ with finite norm $\|X\| := \sqrt{\sum_{j=1}^{\infty} |x_j|^2}$. Show this space is complete. [This is Hilbert’s Hilbert Space]