Problem Set 0: Rust Remover

Due: These problems will not be collected.

You should already have the techniques to do these problems, although they may take some thinking.

1. Show that for any positive integer \( n \), the number \( 2^{n+2} + 3^{2n+1} \) is divisible by 7.

2. Say you have \( k \) linear algebraic equations in \( n \) variables; in matrix form we write \( AX = Y \). Give a proof or counterexample for each of the following.
   a) If \( n = k \) there is always at most one solution.
   b) If \( n > k \) you can always solve \( AX = Y \).
   c) If \( n > k \) the nullspace of \( A \) has dimension greater than zero.
   d) If \( n < k \) then for some \( Y \) there is no solution of \( AX = Y \).
   e) If \( n < k \) the only solution of \( AX = 0 \) is \( X = 0 \).

3. Let \( A \) and \( B \) be \( n \times n \) matrices with \( AB = 0 \). Give a proof or counterexample for each of the following.
   a) \( BA = 0 \)
   b) Either \( A = 0 \) or \( B = 0 \) (or both).
   c) If \( \det A = -3 \), then \( B = 0 \).
   d) If \( B \) is invertible then \( A = 0 \).
   e) There is a vector \( V \neq 0 \) such that \( BAV = 0 \).

4. Let \( A \) be a matrix, not necessarily square. Say \( V \) and \( W \) are particular solutions of the equations \( AV = Y_1 \) and \( AW = Y_2 \), respectively, while \( Z \neq 0 \) is a solution of the homogeneous equation \( AZ = 0 \). Answer the following in terms of \( V \), \( W \), and \( Z \).
   a) Find some solution of \( AX = 3Y_1 \).
   b) Find some solution of \( AX = -5Y_2 \).
   c) Find some solution of \( AX = 3Y_1 - 5Y_2 \).
   d) Find another solution (other than \( Z \) and \( 0 \)) of the homogeneous equation \( AX = 0 \).
   e) Find two solutions of \( AX = Y_1 \).
f) Find another solution of $AX = 3Y_1 - 5Y_2$.

g) If $A$ is a square matrix, then $\det A = ?$

h) If $A$ is a square matrix, for any given vector $W$ can one always find at least one solution of $AX = W$? Why?

5. a) If $r (\neq 0)$ is a rational number and $x$ is irrational, show that both $r + x$ and $rx$ are irrational.

b) Prove that there is no rational number whose square is 12.

c) Graph the points $(x, y)$ in the plane $\mathbb{R}^2$ that satisfy $|y - x| > 2$.

6. a) Write the complex number $z = \frac{1}{a + ib}$ in the form $c + id$, where $a$, $b$, $c$ are $d$ are real numbers. Of course assume $a + ib \neq 0$.

b) If $w \in \mathbb{C}$ satisfies $|w| = 1$, show that $1/w = \bar{w}$. [$\mathbb{C}$ is the set of complex numbers.]

7. Let $z, w, v \in \mathbb{C}$ be complex numbers.

a) Show that $|z - w| \leq |z - v| - |v - w|$.

b) Graph the points $z = x + iy$ in the complex plane that satisfy $1 < |z - i| < 2$.

c) Let $z, w \in \mathbb{C}$ be complex numbers with $|z| < 1$ and $|w| = 1$. Show that

$$\frac{|w - z|}{1 - \bar{z}w} = 1.$$ 

8. a) Find a $2 \times 2$ matrix that rotates the plane by $+45$ degrees ($+45$ degrees means 45 degrees counterclockwise).

b) Find a $2 \times 2$ matrix that rotates the plane by $+45$ degrees followed by a reflection across the horizontal axis.

c) Find a $2 \times 2$ matrix that reflects across the horizontal axis followed by a rotation the plane by $+45$ degrees.

d) Find a matrix that rotates the plane through $+60$ degrees, keeping the origin fixed.

e) Find the inverse of each of these maps.

9. Let the continuous function $f(\theta), 0 \leq \theta \leq 2\pi$ represent the temperature along the equator at a certain moment, say measured from the longitude at Greenwich. Show there are antipodal points with the same temperature.
10. A certain function $f(x)$ has the property that $\int_0^x f(t) \, dt = e^x \cos x + C$. Find both $f$ and the constant $C$.

11. If $b \geq 0$, show that for every real $c$ the equation $x^5 + bx + c = 0$ has exactly one real root.

12. Let $p(x) := x^3 + cx + d$, where $c$, and $d$ are real. Under what conditions on $c$ and $d$ does this has three distinct real roots? [HINT: Sketch a graph of this cubic. Observe that if there are three distinct real roots then there is a local maximum and the polynomial is positive there. What about a local min?]

13. Prove that the function $\sin x$ is not a polynomial. That is, there is no polynomial

$$p(x) = a_0 + a_1 x + \ldots + a_n x^n$$

with real coefficients so that $\sin x = p(x)$ for all real numbers $x$. In your proof you can use any of the standard properties of the function $\sin x$.

[Last revised: August 26, 2010]