Problem Set 1
DUE: Thurs. Sept. 16, 2010. Late papers will be accepted until 1:00 PM Friday.

1. Let \( x_0 = 1 \) and define \( x_k := 3x_{k-1} + 4, \ k = 1, 2, \ldots \) Show that \( x_k < 4 \) and that the \( x_k \) are increasing.

2. Show that \( 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{k!} < 3 \).

3. Let \( A := (a_{ij}) \) be a \( n \times n \) matrix of complex numbers with \( |a_{ij}| \leq M \) and let \( a_{ij}^{(k)} \) be the elements of \( A^k, \ k = 1, 2, \ldots \). Find an estimate for \( |a_{ij}^{(k)}| \) in terms of \( k, n, \) and \( M \).

4. a) Let \( z, w, v \in \mathbb{C} \) and define \( d(z, w) := \frac{|z - w|}{1 + |z - w|} \). Show that
   \[
   d(z, v) \leq d(z, w) + d(w, v)
   \] [triangle inequality].

   b) Let \( S \) be an arbitrary set with \( p, q, r \in S \). Say there is a function \( g : S \times S \to \mathbb{R} \) that satisfies the triangle inequality
   \[
   g(p, r) \leq g(p, q) + g(q, r).
   \]
   Define \( d(p, q) := \frac{g(p, q)}{1 + g(p, q)} \). Show that this function \( d(p, q) \) also satisfies the triangle inequality.

5. Suppose \( a \in \mathbb{R}^k, \ b \in \mathbb{R}^k, \) and \( x \in \mathbb{R}^k \). Find all \( c \in \mathbb{R}^k \) and \( r > 0 \) (depending on \( a \) and \( b \)) such that \( |x - a| = 2|x - b| \) is satisfied if and only if \( |x - c| = r \).

   As an alternate, you may prefer the following generalization. For real \( \lambda > 0, \lambda \neq 1 \), consider the points \( x \in \mathbb{R}^k \) that satisfy
   \[
   |x - a| = \lambda|x - b|.
   \]
   Show that these points lie on a sphere. Part of this is to find the center and radius of this sphere in terms of \( a, b \) and \( \lambda \). What if \( \lambda = 1 \)?

[Last revised: September 18, 2010]