Problem Set 10
DUE: Tues. Nov. 30, 2010. Late papers will be accepted until 1:00 PM Wednesday.

Note: We say a function is smooth if its derivatives of all orders exist and are continuous.

1. Find an integer \( N \) so that \( 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{N} > 100 \).

2. Let \( c(x) \) be a given smooth function and \( u(x) \neq 0 \) satisfy the differential equation \(-u'' + c(x)u = \lambda u\) on the bounded interval \( \Omega = \{a < x < b\} \) with \( u = 0 \) on the boundary of \( \Omega \). Here \( \lambda \) is a constant. Show that
\[
\lambda = \frac{\int_{\Omega} (u'^2 + cu^2) \, dx}{\int_{\Omega} u^2 \, dx}
\]

3. The Gamma function is defined by \( \Gamma(x) := \int_0^\infty e^{-t} t^{x-1} \, dt \).
   a) For which real \( x \) does this improper integral converge?
   b) Show that \( \Gamma(x+1) = x\Gamma(x) \) and deduce that \( \Gamma(n+1) = n! \) for any integer \( n \geq 0 \).

4. Consider \( f(x) := \sum_{k=1}^\infty \frac{\sin(kx)}{1+k^4} \).
   a) For which real \( x \) is \( f \) continuous?
   b) Is \( f \) differentiable? Why?

5. If the complex power series \( \sum_{k=0}^\infty a_k z^k \) converges at \( z = c \), and \( R < |c| \), show that it converges absolutely and uniformly in the disk \( \{z \in \mathbb{C} \mid |z| \leq R\} \).

6. Let \( a_n \) be a bounded sequence of complex numbers and
\[
f(z) = \sum_{n=1}^\infty \frac{a_n}{n^z},
\]
where \( z = x + iy \). If \( c > 1 \), show that this series converges absolutely and uniformly in the half-plane \( \{z = x + iy \in \mathbb{C} \mid x \geq c\} \).

7. Show that the sequence of functions \( f_n(x) := n^3 x^n (1-x) \) does not converge uniformly on \([0, 1]\).
8. For each of the following give an example of a sequence of continuous functions. Justify your assertions. [A clear sketch may be adequate — as long as it is convincing].

a) \( f_n(x) \) that converge to zero at every \( x, \ 0 \leq x \leq 1 \), but not uniformly.

b) \( g_n(x) \) that converge to zero at every \( x, \ 0 \leq x \leq 1 \), but \( \int_0^1 g_n(x) \, dx \geq 1 \).

c) \( h_n(x) \) converge to zero uniformly for \( 0 \leq x < \infty \), but \( \int_0^\infty h_n(x) \, dx \geq 1 \).

Bonus Problems (Due Nov 30)

B-1 If \( \varphi: \mathbb{R} \to \mathbb{R} \) is continuous with period \( P \), so \( \varphi(x+P) = \varphi(x) \) for all real \( x \). Show that

\[
\lim_{\lambda \to \infty} \int_0^1 f(x)\varphi(\lambda x) \, dx = \int_0^1 f(x) \, dx,
\]

where \( \overline{\varphi} := \frac{1}{P} \int_0^P \varphi(t) \, dt \) is the average of \( \varphi \) over one period. [This generalized both HW8 #7 and HW9 #B-1.]

B-2 Let \( \varphi_n(t) \) be a sequence of smooth real-valued functions with the properties

(a) \( \varphi_n(t) \geq 0 \),  
(b) \( \varphi_n(t) = 0 \) for \( |t| \geq 1/n \),  
(c) \( \int_{-\infty}^{\infty} \varphi_n(t) \, dt = 1 \).

Note: because of (b), this integral is only over \(-1/n \leq t \leq 1/n\).

Assume \( f(x) \) is uniformly continuous for all \( x \in \mathbb{R} \) and define

\[
f_n(x) := \int_{-\infty}^{\infty} f(x-t)\varphi_n(t) \, dt.
\]

Show that \( f_n(x) \) converges uniformly to \( f(x) \) for all \( x \in \mathbb{R} \). [SUGGESTION: Use \( f(x) = f(x) \left( \int_{-\infty}^{\infty} \varphi_n(t) \, dt \right) = \int_{-\infty}^{\infty} f(x)\varphi_n(t) \, dt \). Also, note explicitly where you use the uniform continuity of \( f \).]

REMARK: One can show that the approximations \( f_n \) are also smooth. Thus, this proves that you can approximate a continuous function uniformly on any compact set by a smooth function.

[Last revised: November 21, 2010]