Problem Set 7
DUE: Thurs. Nov. 4, 2010. Late papers will be accepted until 1:00 PM Friday.

Note: We say a function is smooth if its derivatives of all orders exist and are continuous.

1. Let $f: [a, \infty) \to \mathbb{R}$ be a smooth function whose first derivative is bounded: $|f'(x)| \leq M$ for all $x \geq a$. Prove that it is uniformly continuous on $[a, \infty)$.

As immediate examples, $x^{1/3}$ is uniformly continuous for all $x \geq 1$ and $\cos x$ is uniformly continuous for all $x$.

2. a) Show that $\sin x$ is not a polynomial.

b) Show that $\sin x$ is not a rational function, that is, it cannot be the quotient of two polynomials.

c) Let $v(t)$ be periodic with period 1, so $v(t + 1) = v(t)$ for all real $t$. If $v$ is not a constant, show that it cannot be a rational function. that is, $v$ cannot be the quotient of two polynomials.

d) Show that $e^x$ is not a rational function.

3. a) If a smooth function $f(x)$ has the property that $f''(x) \geq 0$ for all $x$, show that it is convex, that is, at every point the graph of the curve $y = f(x)$ lies above all its tangent lines.

b) Let $v(x)$ be a smooth real-valued function for $0 \leq x \leq 1$. If $v(0) = v(1) = 0$ and $v''(x) > 0$ for all $0 \leq x \leq 1$, show that $v(x) \leq 0$ for all $0 \leq x \leq 1$.

c) Prove that the function $e^x$ is convex.

d) Show that $e^x \geq 1 + x$ for all real $x$.

4. a) Let $p(x) := x^3 + cx + d$, where $c$, and $d$ are real. Under what conditions on $c$ and $d$ does this have three distinct real roots? [ANSWER: $c < 0$ and $d^2 < -4c^3 / 27$].

b) Generalize to the real polynomial $p(x) := ax^3 + bx^2 + cx + d$ ($a \neq 0$) by a change of variable $t = x - \alpha$ (with a clever choice of $\alpha$) to reduce to the above special case.

5. Let

$$p_n(x) := \left( \frac{d}{dx} \right)^n (1 - x^2)^n.$$

This is a polynomial of degree $n$. Show that it has $n$ real distinct zeroes, all in the interval $-1 < x < 1$. 

6. a) For any integer \( n \geq 0 \), show that \( \lim_{x \searrow 0} e^{-1/x} x^n = 0 \).

b) Define \( f : \mathbb{R} \rightarrow \mathbb{R} \) by
\[
f(x) = \begin{cases} 
  e^{-\frac{1}{x}} & \text{for } x > 0, \\
  0 & \text{for } x \leq 0.
\end{cases}
\]
Sketch the graph of \( f \).

c) Show that \( f \) is a smooth function for all real \( x \).

d) Show that each of the following are smooth and sketch their graphs:
\[
\begin{align*}
g(x) &= f(x) f(1-x) \\
k(x) &= h(x) h(4-x) \\
\varphi(x,y) &= K(x) K(y), \ (x,y) \in \mathbb{R}^2
\end{align*}
\]
\[
\begin{align*}
h(x) &= \frac{f(x)}{f(x) + f(1-x)} \\
K(x) &= k(x+2), \\
\Phi(x) &= K(\|x\|), \ x = (x_1, x_2) \in \mathbb{R}^2
\end{align*}
\]

Bonus Problem (Due Nov 4)

B-1 [Interpolation] Let \( x_0 < x_1 < x_2 \) be distinct real numbers and \( f(x) \) a smooth function.

a) Show there is a unique quadratic polynomial \( p(x) \) with the property that \( p(x_j) = f(x_j) \) for \( j = 0, 1, 2 \).

b) [Remainder term in interpolation] If \( b \) is in the open interval \( (x_0, x_2) \) with \( b \neq x_j, \ j = 0, 1, 2 \), show there is a point \( c \) (depending on \( b \)) in the interval \( (x_0, x_2) \) so that
\[
f(b) = p(b) + \frac{f'''(c)}{3!} (b-x_0)(b-x_1)(b-x_2).
\]

[SUGGESTION: Define the constant \( M \) by
\[
f(b) = p(b) + M(b-x_0)(b-x_1)(b-x_2),
\]
and look at
\[
g(x) := f(x) - [p(x) + M(x-x_0)(x-x_1)(x-x_2)].
\]

B-2 In \( \ell_1 \) let \( S \) be the set of unit vectors \( e_1 = (1,0,0,\ldots), \ e_2 = (0,1,0,0,0,\ldots) \) etc. This set \( S \) is both closed and bounded. Give an example of a continuous \( f : S \rightarrow R \) that is bounded from above, say \( \sup_{x \in S} f(x) = c \) but there is no point \( p \in S \) where \( f(p) = c \).

[MORAL: In a normed linear space the condition of a set being compact is much stronger than just closed and bounded.

[Last revised: October 28, 2010]