My signature below certifies that I have complied with the University of Pennsylvania's Code of Academic Integrity in completing this exam.

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Signature PRINTED NAME

Math 508 **Exam 1** Jerry Kazdan
October 14, 2014 9:00 – 10:20

DIRECTIONS This exam has three parts. Part A asks for 8 examples (3 points each, so 24 points), Part B has 4 shorter problems, (8 points each so total 32 points) while Part C had 3 problems (15 points each, so total is 45 points). Maximum total score is thus 101 points. Closed book, no calculators or computers– but you may use one $3'' \times 5''$ card with notes on both sides.

Remember to silence your cellphone before the exam and keep it out of sight for the duration of the test period. This exam begins promptly at 9:00 and ends at 10:20pm; anyone who continues working after time is called may be denied the right to submit his or her exam or may be subject to other grading penalties. Please indicate what work you wish to be graded and what is scratch. *Clarity and neatness count.*

PART A: For each of the following give an example of a subset with the specified properties. [3 points each, total 24 points]

Score	
A-1 — A4	
A-5 — A8	
B-1	
B-2	
B-3	
B-4	
C-1	
C-2	
C-3	
Total	

A–1. A closed subset of $\mathbb{R}^2$ that is not compact,

A–2. An open subset of $\mathbb{R}^2$ that is disconnected,

A–3. Bounded set in $\mathbb{R}^2$ with exactly two limit points.

A–4. An open cover of $\{x \in \mathbb{R} \ : \ 0 < x \leq 1\}$ that has no finite sub-cover.

A–5. A metric space X having some bounded infinite sequence with no subsequence converging to a point in X.

A–6. A metric space that is not complete.

A–7. A series of complex numbers $\sum_1^\infty a_k$ where the corresponding sequence of partial sums $S_n = \sum_1^n a_k$ is bounded but the series *diverges*.

A–8. A closed and bounded set E in a complete metric space X with E *not* compact.

PART B BEGINS ON THE NEXT PAGE

PART B Four shorter problems, 8 points each (so 32 points total).

B–1. Let A_k and B_k, $k = 1, 2, 3, \ldots$ be sequences of $n \times n$ matrices. If $A_k \to A$ and $B_k \to B$, prove (using ϵ and N) that $A_k B_k \to AB$.

B–2. Show that a compact set K in a metric space is bounded.

B–3. Find the supremum and infimum of the set B defined below. Then find the closure of B.

$$B := \left\{ \frac{n^2+2}{n^2+1} : n = 0, 1, 2, \ldots \right\}.$$

Please justify your assertions.

B–4. Let $K_j,\ j = 1, 2, \ldots$ be compact sets in a metric space. Give a proof or counterexample for each of the following assertions.

a) $K_1 \cap K_2$ is compact.

b) $K_1 \cup K_2$ is compact.

c) $\bigcup_{j=1}^{\infty} K_j$ is compact.

PART C Three questions, 15 points each (so 45 points total).

C–1. Let $\{a_k\} \in \mathbb{R}$ be a sequence of real numbers. If a_k converges to some *positive* $A > 0$, show there is an integer N so that if $n > N$, then $a_n > 0$.

C–2. Let $\{a_n\} \in \mathbb{C}$ be a *contracting* sequence, that is there is a $0 < c < 1$ so that

$$|a_{n+1} - a_n| \le c|a_n - a_{n-1}|, \quad n = 1, 2, 3, \ldots.$$

a) Show that $\quad |a_{n+1} - a_n| \le c^n |a_1 - a_0|.$

b) If $n > k$, show that $\quad |a_n - a_k| \le \dfrac{c^k}{1-c}|a_1 - a_0|.$

c) Show that the sequence a_n converges.

C–3. Say the complex power series $\sum_{n=0}^{\infty} a_n z^n$ converges at a point $z_0 \in \mathbb{C}$. If $|z| < |z_0|$, show that $\sum_{n=1}^{\infty} n a_n z^{n-1}$ converges absolutely. [There are several different ways to do this.]