Math 508 **Exam 1** Jerry Kazdan
October 14, 2014 9:00 – 10:20

DIRECTIONS This exam has three parts. Part A asks for 8 examples (3 points each, so 24 points), Part B has 4 shorter problems, (8 points each so total 32 points) while Part C had 3 problems (15 points each, so total is 45 points). Maximum total score is thus 101 points. Closed book, no calculators or computers– but you may use one $3'' \times 5''$ card with notes on both sides.

Remember to silence your cellphone before the exam and keep it out of sight for the duration of the test period. This exam begins promptly at 9:00 and ends at 10:20pm; anyone who continues working after time is called may be denied the right to submit his or her exam or may be subject to other grading penalties. Please indicate what work you wish to be graded and what is scratch. *Clarity and neatness count.*

PART A: For each of the following give an example of a subset with the specified properties. [3 points each, total 24 points]

A–1. A closed subset of $\mathbb{R}^2$ that is not compact,

SOLUTION: Many many examples. i). All of R^2, ii). $\{(x,y)\in\mathbb{R}^2 : y=0\}$
iii). $\{(x,y)\in\mathbb{R}^2 : 1\le y\le 7\}$ iv). $\{(x,y)\in\mathbb{R}^2 : 1\le x+y\le 7\}$
v). Any closed set in $\mathbb{R}^2$ that is not bounded.

A–2. An open subset of $\mathbb{R}^2$ that is disconnected,

SOLUTION: $\{(x,y)\in\mathbb{R}^2 : x<0 \text{ or } x>0\}$

A–3. Bounded set in $\mathbb{R}^2$ with exactly two limit points.

SOLUTION: $\{(x_n,\,0)\in\mathbb{R}^2 : x_n=(-1)^n+\frac{1}{n}\}$. The limit points are $(\pm1,\,0)$.

A–4. An open cover of $\{x\in\mathbb{R} \;:\; 0<x\le 1\}$ that has no finite sub-cover.

SOLUTION: $U_n=\{x\in\mathbb{R} : \frac{1}{n}<x<2,\; n=1,\,2,\,\ldots\}$

A–5. A metric space X having some bounded infinite sequence with no subsequence converging to a point in X.

SOLUTION: i). The set $\{1,\frac{1}{2},\frac{1}{3},\ldots,\frac{1}{n},\ldots\}$.
ii). The set X of rational numbers in $\{0\le x<\sqrt{2}\}$ where the sequence is rational numbers converging to $\sqrt{2}$ (note $\sqrt{2}$ is not in this set).

A–6. A metric space that is not complete.

SOLUTION: i). The rational numbers in $\mathbb{R}$.
ii). Any subset of $\mathbb{R}$ that is not closed.

A–7. A series of complex numbers $\sum_1^\infty a_k$ where the corresponding sequence of partial sums $S_n = \sum_1^n a_k$ is bounded but the series *diverges*.

SOLUTION: The series $\sum_{n=0}^\infty (-1)^n$, so $a_k = (-1)^k$.

A–8. A closed and bounded set E in a complete metric space X with E *not* compact.

SOLUTION: This is the only of these example that is not really elementary. Let ℓ_2 be the set of all real infinite sequences $X = (x_1, x_2, x_3, \ldots)$ with $\sum_{n=1}^\infty x_j^2 < \infty$ and let $e_1 = (1, 0, 0, \ldots)$, $e_2 = (0, 1, 0, 0, \ldots)$, $e_3 = (0, 0, 1, 0, 0, \ldots)$, $e_4 = \ldots$ be the standard unit vectors.

Then the set $E = \{e_1, e_2, e_3, \ldots, \}$ is closed and bounded but *not* compact since $|e_i - e_j| = \sqrt{2}$ so there can't be a convergent subsequence.

Alternately, E is covered by the open balls B_j centered at e_j and radius $1/2$, $j = 1, 2, 3, \ldots$. This open cover has no finite sub-cover.

PART B Four shorter problems, 8 points each (so 32 points total).

B–1. Let A_k and B_k, $k = 1, 2, 3, \ldots$ be sequences of $n \times n$ matrices. If $A_k \to A$ and $B_k \to B$, prove (using ϵ and N) that $A_k B_k \to AB$.

SOLUTION: This is identical to the case where A_k and B_k are real numbers – except that for matrices one should not presume that $AB = BA$ – since it is usually *false*.

We use the triangle inequality and the properties of the norm we defined on matrices. In articular, $|AB| \le |A||B|$. Then

$$\begin{aligned} |A_k B_b - AB| &= |(A_k - A)B_k + A(B_k - B)| \\ &\le |(A_k - A)B_k| + |A(B_k - B)| \\ &\le |A_k - A||B_k| + |A||B_k - B|. \end{aligned}$$

Since $B_k \to B$, the sequence B_k is bounded, that is, there is a real number $c > 0$ such that $|B_k| \le c$ for all k.

Also, given any $\epsilon_1 > 0$ and $\epsilon_2 > 0$ there are positive integers N_1 and N_2 so that if $n > N_1$ then $|A_n - A| < \epsilon_1$ while if $n > N_2$ then $|B_n - B| < \epsilon_2$. Thus, if $N \ge \max\{N_1, N_2\}$ then

$$|A_n B_n - AB| \le \epsilon_1 c + |A|\epsilon_2.$$

Since we cand choose any ϵ_1 and ϵ_2, pick $\epsilon_1 < \epsilon/(2c)$ and, if $A \ne 0$, pick $\epsilon_2 < \epsilon/(2|A|)$ (if $A = 0$ we don't need the ϵ_2 term). This gives the desired

$$|A_n B_n - AB| < \epsilon$$

The computation is slightly shorter if one makes the preliminary substitution $P_n := A_n - A \to 0$, and $Q_n := B_n - B \to 0$. Then

$$A_nB_n - AB = P_nQ_n + P_nB + AQ_n$$

B–2. Show that a compact set K in a metric space is bounded.

SOLUTION: Let p be a point in the space and let B_k, $k = 1, 2, 3, \ldots$ be the open balls centered at p with radius k. These cover the whole metric space, in particular they cover the compact set. Since the set is compact, a finite subset of these balls covers K. Then K is in the largest of these balls.

[Instead of covering by balls, one can cover by any *bounded* open sets. – but it is simplest to use balls.]

B–3. Find the supremum and infimum of the set B defined below. Then find the closure of B.

$$B := \left\{ \frac{n^2+2}{n^2+1} : n = 0, 1, 2, \ldots \right\}.$$

Please justify your assertions.

SOLUTION: Since $x_n := \dfrac{n^2+2}{n^2+1} = 1 + \dfrac{1}{n^2+1}$, the sequence $\{x_n\}$is clearly decreasing. It's supremum (and maximum) is clearly 2 (let $n = 0$) while its infimum is 1 (let $n \to \infty$). It has no minimum.

The closure of this set is $B \cup \{1\}$ (it is *not* the interval $\{1 \le x \le 2\}$).

B–4. Let K_j, $j = 1, 2, \ldots$ be compact sets in a metric space. Give a proof or counterexample for each of the following assertions.

a) $K_1 \cap K_2$ is compact.

SOLUTION: Since K_1 and K_2 are compact, they are closed, so their intersection is also closed. This intersection is a closed subset of the compact set K_1 and hence is compact.

As done in the next part, a direct proof using open covers of K_1 and K_2 is equally simple.

b) $K_1 \cup K_2$ is compact.

SOLUTION: Let $\{U_i\}$ be an open cover of K_1 and $\{V_J\}$ be an open cover of K_2. Then their union $\{U_i\} \cup \{V_j\}$ is an open cover of $K_1 \cup K_2$. By compactness, a finite subset of the $\{U_i\}$ covers K_1 and a finite subset of the $\{V_j\}$ covers K_2, then the union of these two finite covers is a finite cover of $K_1 \cup K_2$.

c) $\bigcup_{j=1}^{\infty} K_j$ is compact.

SOLUTION: Counterexamples.
i). In $\mathbb{R}$ let $K_j = \{x \in \mathbb{R} : \frac{1}{j} \le x \le 1,\ j = 1, 2, \ldots\}$. These are closed and bounded sets in $\mathbb{R}$ and hence compact. But $\cup_j K_j = (0,\ 1]$ is not closed and hence not compact.
ii). In $\mathbb{R}$ let $K_j = \{x \in \mathbb{R} : -j \le x \le j,\ j = 1, 2, \ldots\}$. These are closed bounded sets in $\mathbb{R}$, and hence compact. But $\cup_j K_j = \mathbb{R}$ is not bounded and hence not compact.

PART C Three questions, 15 points each (so 45 points total).

C–1. Let $\{a_k\} \in \mathbb{R}$ be a sequence of real numbers. If a_k converges to some *positive* $A > 0$, show there is an integer N so that if $n > N$, then $a_n > 0$.

SOLUTION: Pick $\epsilon = A/2$. Then there is an integer N so that if $n > N$ then $|a_n - A| < \epsilon$, that is,

$$-\epsilon < a_n - A < \epsilon,$$

In particular, $A - \epsilon < a_n$. Using $\epsilon = A/2$, then for all $n > N$ we have $0 < \frac{1}{2}A < a_n$.

C–2. Let $\{a_n\} \in \mathbb{C}$ be a *contracting* sequence, that is there is a $0 < c < 1$ so that

$$|a_{n+1} - a_n| \le c|a_n - a_{n-1}|, \quad n = 1, 2, 3, \ldots.$$

a) Show that $|a_{n+1} - a_n| \le c^n |a_1 - a_0|$.

SOLUTION: Clearly

$$|a_4 - a_3| \le c|a_3 - a_2| \le c^2|a_2 - a_1| \le c^3|a_1 - a_0|.$$

Repeating this the assertion is obvious.

b) If $n > k$, show that $|a_n - a_k| \le \dfrac{c^k}{1-c}|a_1 - a_0|$.

SOLUTION: Say $n > k$. Then by the triangle inequality and part (a),

$$\begin{aligned} |a_n - a_k| =& |(a_n - a_{n-1}) + (a_{n-1} - a_{n-2}) + \cdots + (a_{k+1} - a_k)| \\ \le& |a_n - a_{n-1}| + |a_{n-1} - a_{n-2}| + \cdots + |a_{k+1} - a_k| \\ \le& \left(c^{n-1} + c^{n-2} + \cdots + c^k\right) |a_1 - a_0| \\ =& c^k \left(\frac{1 - c^n}{1 - c}\right) |a_1 - a_0| < \frac{c^k}{1-c}|a_1 - a_0|. \end{aligned}$$

c) Show that the sequence a_n converges.

SOLUTION: Since $0 < c < 1$ then given any $\epsilon > 0$, for k sufficiently large $c^k < \epsilon$. Thus by the previous part, if $n > k$, then

$$|a_n - a_k| < \frac{|a_1 - a_0|}{1 - c}\epsilon.$$

This shows that the $\{a_k\}$ is a Cauchy sequence. Because $\mathbb{R}$ is complete, there is some real number A to which the $\{a_n\}$ converges.

C–3. Say the complex power series $\sum_{n=0}^{\infty} a_n z^n$ converges at a point $z_0 \neq 0 \in \mathbb{C}$. If $|z| < |z_0|$, show that $\sum_{n=1}^{\infty} n a_n z^{n-1}$ converges absolutely. [There are several different ways to do this.]

SOLUTION 1:

$$\sum_{n=1}^{\infty} |n a_n z^{n-1}| = \sum_{n=1}^{\infty} \frac{n|a_n z_0^n|}{|z_0|} \left|\frac{z}{z_0}\right|^{n-1}. \tag{1}$$

Because $\sum a_n z_0^n$ converges, then $a_n z_0^n \to 0$. Thus the sequence $|a_n z_0^n|$ is bounded, say $|a_n z_0^n| \le M$ for all n.. Condequently equation (1) gives

$$\sum_{n=1}^{\infty} |n a_n z^{n-1}| \le \frac{M}{|z_0|} \sum_{n=1}^{\infty} n \left|\frac{z}{z_0}\right|^{n-1}.$$

Since $|z/z_0| < 1$, this last series converges by the ratio test.

SOLUTION 1′: This is just a small reorganization of the solution just above. Since $\sum_{n=0}^{\infty} a_n z_0 n$ converges, then $a_n z_0^n \to 0$. Consequently this sequence is bounded, that is, for some M we have $|a_n z_0^n| \le M$ for all n. This gives the inequality

$$|a_n| \le \frac{M}{|z_0|^n}.$$

Therefore

$$|n a_n z^{n-1}| \le \frac{nM}{|z_0|} \left|\frac{z}{z_0}\right|^{n-1}.$$

Because $|z/z_0| < 1$, by the ratio test the series $\sum n|\frac{z}{z_0}|^{n-1}$ converges. Therefore, by the comparison test $\sum_{n=1}^{\infty} n a_n z^{n-1}$ converges absolutely.

SOLUTION 2: By a standard theorem (see Rudin, p. 69, 3.39), for the power series $\sum c_n z^n$, let

$$\alpha = \limsup_{n\to\infty} |c_n|^{1/n}.$$

Then the radius of the disk of convergence is $R := 1/\alpha$. Also, inside this circle, so $|z| < R$, the power series converges absolutely. For $\sum a_n z^n$, the radius R_1 is thus

$$\frac{1}{R_1} = \limsup_{n\to\infty} |a_n|^{1/n}$$

while for $\sum na_n z^{n-1}$ since $n^{1/n} \to 1$, the radius, R_2, is

$$\frac{1}{R_2} = \limsup_{n\to\infty} |n\, a_n|^{1/n} = \limsup_{n\to\infty} |a_n|^{1/n} = \frac{1}{R_1}.$$

Thus $R_2 = R_1$. This proof assumed $R_1 \neq 0$. However if $R_1 = 0$ the same reasoning shows that $R_2 = 0$.