My signature below certifies that I have complied with the University of Pennsylvania's Code of Academic Integrity in completing this exam.

Signature PRINTED NAME

Math 508
December 9, 2014

Exam 2

Jerry L. Kazdan
9:00 – 10:20

DIRECTIONS This exam has three parts. Part A has 8 True/False question (2 points each so total 16 points), Part B has 5 shorter problems (6 points each, so 30 points), while Part C has 5 traditional problems (12 points each, so total is 60 points). Maximum score is thus 106 points.

Closed book, no calculators or computers– but you may use one $3'' \times 5''$ card with notes on both sides.

Please remember to silence your cellphone before the exam and keep it out of sight for the duration of the test period. This exam begins promptly at 9:00 and ends at 10:20; anyone who continues working after time is called may be denied the right to submit his or her exam or may be subject to other grading penalties. Please indicate what work you wish to be graded and what is scratch. *Clarity and neatness count.*

Score	
A-1 — A-8	
B-1	
B-2	
B-3	
B-4	
B-5	
C-1	
C-2	
C-3	
C-4	
C-5	
Total	

Part A 8 True/False questions (2 points each, so 16 points). Answer only, no reasons need be given. *Circle* your True/False choice. [If you wish to give a reason or example, your work will be read.]

A–1. **T F** The quotient of two irrational numbers (with the denominator nonzero) is irrational.

A–2. **T F** If A is a nonempty compact subset of the real line $\mathbb{R}$, then $\mathbb{R} - A$ is never connected. (The set $R - A$ consists of all real numbers which are not in A.)

A–3. **T F** If A and B are compact subsets of a metric space, then $A \cup B$ is also compact.

A–4. **T F** A series of complex numbers converges if and only if the corresponding sequence of partial sums is bounded.

A–5. **T F** If X is any metric space and $f : X \to \mathbb{R}$ is any continuous real-valued function, then the function $g : X \to \mathbb{R}$ defined by $g(x) = (f(x))^2$ is always continuous.

A–6. **T F** Let X and Y be metric spaces, and let A and B be two subsets of X whose union is X. If $f : X \to Y$ is continuous on A and continuous on B, then it is continuous on X.

A–7. **T F** If $f : X \to Y$ is a continuous map between metric spaces, and $f(X)$ is compact, then X is compact.

A–8. **T F** A closed and bounded subset of a metric space must be compact.

Part B 5 shorter problems (6 points each, so 30 points)

B–1. For each of the following give an example of a sequence of continuous functions. If you prefer, a clear sketch of a graph will be adequate.

a) $f_n(x) \to 0$ for all $x \in [0,\ 1]$ but $\displaystyle\int_0^1 f_n(x)\,dx \geq 1$ for all $n = 1, 2, \ldots$.

b) $g_n(x) \to 0$ for all $x \in [0,\ 1]$ and $\displaystyle\int_0^1 g_n(x)\,dx \to 0$ but the g_n do not converge uniformly to zero on $[0,\ 1]$.

B–2. Show that there is some real $x > 1$ so that $\dfrac{x^2+5}{3+x^6} = 1$.

B–3. Say $\int_0^x f(t)\,dt = \sin(1+x^2)+C$, assuming that $f(t)$ is continuous and C is a constant, find both C and f.

B–4. Let $f(x) = \displaystyle\sum_{n=2}^{\infty} \frac{1}{n^2+\cos nx}$

a) Prove that the series converges uniformly for all real x.

b) Where (if anywhere) is f continuous? Why?

B–5. Let $f\colon \mathbb{R} \to \mathbb{R}$ be a smooth function with the properties that $f''(x) \geq 0$ and $f(x) \leq$ const for all $x \in \mathbb{R}$. Show that $f(x) =$ constant.

Part C 5 traditional problems (12 points each, so total is 60 points)

C–1. Let a_n be a sequence of complex numbers with $a_n \to A$ as $n \to \infty$ and let

$$S_n = \frac{a_1 + a_2 + \cdots + a_n}{n}.$$

Show that $S_n \to A$ as $n \to \infty$.

C–2. Let $f(x) \in C([0,1])$ have the property that $\int_0^1 f(x)h(x)\,dx = 0$ for it all functions $h \in C([0,1])$ with the additional property that $h(0) = h(1) = 0$. Prove that $f(x) \equiv 0$ on all of [0, 1].

C–3. Let $\mathcal{M}$ be a metric space and let $B(\mathcal{M};\mathbb{R})$ be the set of all bounded real-valued functions on $\mathcal{M}$ with the uniform norm:

$$\|f\| = \sup_{x\in\mathcal{M}} |f(x)|.$$

Since f is assumed to be bounded, then $\|f\| < \infty$. Define the distance between f and g to be $\|f - g\|$. This makes B into a metric space. Show that this is a *complete* metric space.

There are two steps: (i). Get a candidate for the limit function $f(x)$, and (ii). Prove that this $f(x)$ is bounded. [Where does your proof use that $\mathbb{R}$ is complete?]

C–4. Suppose that $G : \mathbb{R}^n \to \mathbb{R}^n$ is a continuous function with the property that for some real M

$$\|G(x) - G(y)\| \le M\|x - y\| \qquad \text{for all } \; x, y \in \mathbb{R}^n. \tag{1}$$

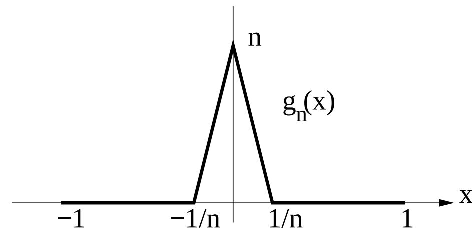
Here $\|x\|$ is the standard Euclidean distance in $\mathbb{R}^n$.

If $\lambda > 0$ is small enough, show that the function $F : \mathbb{R}^n \to \mathbb{R}^n$ defined by

$$F(x) = x - \lambda G(x)$$

is one-to-one and onto, so for every $z \in \mathbb{R}^n$ the equation $F(x) = z$ has one and only one solution $x \in \mathbb{R}^n$. Note that a solution x is a fixed point of some map.

C–5. Let $f(x) : \mathbb{R} \to \mathbb{R}$ be continuous for all x with $f(x) = 0$ for $|x| \geq 1$ and let $g_n(x)$ be the sequence of functions in the figure. Let



$$h_n(t) = \int_{-1}^{1} f(t-x) g_n(x)\, dx$$

a) Show that h_n is uniformly continuous.

b) Show that $\lim_{n\to\infty} h_n(t) = f(t)$ uniformly.