Math 508 **Exam 2** Jerry L. Kazdan
December 9, 2014 9:00 – 10:20

DIRECTIONS This exam has three parts. Part A has 8 True/False question (2 points each so total 16 points), Part B has 5 shorter problems (6 points each, so 30 points), while Part C has 5 traditional problems (12 points each, so total is 60 points). Maximum score is thus 106 points.

Closed book, no calculators or computers– but you may use one $3'' \times 5''$ card with notes on both sides.

Please silence your cellphone before the exam and keep it out of sight for the duration of the test period. This exam begins promptly at 9:00 and ends at 10:20.
Anyone who continues working after time is called may be denied the right to submit his or her exam or may be subject to other grading penalties. Please indicate what work you wish to be graded and what is scratch. *Clarity and neatness count.*

Part A 8 True/False questions (2 points each, so 16 points). Answer only, no reasons need be given. *Circle* your True/False choice. [If you wish to give a reason or example, your work will be read.]

A–1. **T F** The quotient of two irrational numbers (with the denominator nonzero) is irrational.

SOLUTION FALSE $\sqrt{2}/\sqrt{2}$ or, say, $\sqrt{2}/\sqrt{8}$

A–2. **T F** If A is a nonempty compact subset of the real line $\mathbb{R}$, then $\mathbb{R} - A$ is never connected. (The set $R - A$ consists of all real numbers which are not in A.)

SOLUTION TRUE

A–3. **T F** If A and B are compact subsets of a metric space, then $A \cup B$ is also compact.

SOLUTION TRUE

A–4. **T F** A series of complex numbers converges if and only if the corresponding sequence of partial sums is bounded.

SOLUTION FALSE. For instance the series $\sum(-1)^n$

A–5. **T F** If X is any metric space and $f : X \to \mathbb{R}$ is any continuous real-valued function, then the function $g : X \to \mathbb{R}$ defined by $g(x) = (f(x))^2$ is always continuous.

SOLUTION TRUE The product of two continuous real (or complex-valued) functions is continuous.

A–6. **T F** Let X and Y be metric spaces, and let A and B be two subsets of X whose union is X. If $f : X \to Y$ is continuous on A and continuous on B, then it is continuous on X.

SOLUTION FALSE. Example: Let $A = \{0 \le x < 1\}$, $B = \{1 \le x \le 2\}$, $X = \{0 \le x \le 2\}$ with $f(x) = 0$ on A, and $f(x) = 1$ on B.

A–7. **T F** If $f : X \to Y$ is a continuous map between metric spaces, and $f(X)$ is compact, then X is compact.

SOLUTION FALSE. Example: Say f maps all of X to one point $p \in Y$. A set consisting of one point is compact. This gives no information about X.

A–8. **T F** A closed and bounded subset of a metric space must be compact.

SOLUTION FALSE (although true in $\mathbb{R}^n$). Example: the unit sphere in ℓ_2.

Part B 5 shorter problems (6 points each, so 30 points)

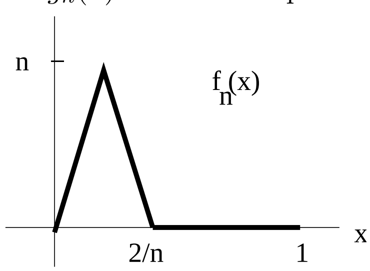
B–1. For each of the following give an example of a sequence of continuous functions $f_n(x) \ge 0$. If you prefer, a clear sketch of a graph will be adequate.

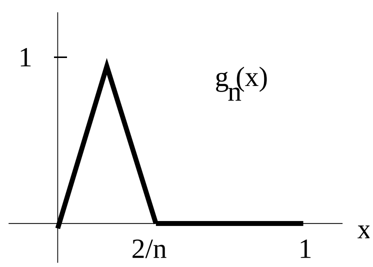
a) $f_n(x) \to 0$ for all $x \in [0, 1]$ but $\int_0^1 f_n(x)\,dx \ge 1$ for all $n = 1, 2, \ldots$.

SOLUTION Let $f_n(x)$ be the "bump" function in the figure on the left below.

b) $g_n(x) \to 0$ for all $x \in [0, 1]$ and $\int_0^1 g_n(x)\,dx \to 0$ but the g_n do not converge uniformly to zero on [0, 1].

SOLUTION Let $g_n(x)$ be the "bump" function in the figure on the right below.





B–2. Show that there is some real $x > 1$ so that $\dfrac{x^2+5}{3+x^6} = 1$.

Solution Let $f(x) = \dfrac{x^2+5}{3+x^7}$. Then $f(1) = (1+5)/(3+1) > 1$ while $f(2) = (4+5)/(3+64) < 1$ so the assertion follows from the Intermediate Value Theorem.

B–3. Say $\int_0^x f(t)\,dt = \sin(1+x^2)+C$, assuming that $f(t)$ is continuous and C is a constant, find both C and f.

Solution Let $x = 0$ on both sides to find that $0 = \sin(1) + C$ so $C = -\sin(1)$.

Take the derivative of both sides to get: $f(x) = 2x\cos(1+x^2)$

B–4. Let $f(x) = \displaystyle\sum_{n=2}^{\infty} \frac{1}{n^2+\cos nx}$

a) Prove that the series converges uniformly for all real x.

Solution Since $|\cos nx| \le 1$, then $n^2 + \cos nx \ge n^2 - 1$ Therefore

$$\frac{1}{n^2+\cos nx} \le \frac{1}{n^2-1}.$$

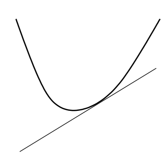
Because the series $\sum_{n=2}^{\infty} \frac{1}{n^2-1}$ converges, by the Weierstrass M test, the original series converges absolutely and uniformly for all x.

b) Where (if anywhere) is f continuous? Why?

Solution Since the uniform limit of a sequence of continuous functions is continuous, this $f(x)$ is continuous for all real x.

B–5. Let $f\colon \mathbb{R} \to \mathbb{R}$ be a smooth function with the properties that $f''(x) \ge 0$ and $f(x) \le$ const. for all $x \in \mathbb{R}$. Show that $f(x) =$ constant.

Solution That $f''(x) \ge 0$ for all x implies the graph of f is convex. Therefore it lies above every tangent line. Say there is a point p where $f'(p) > 0$. Since the graph of f lies above this tangent line at p this would imply that as $x \to \infty$ then $f(x) \to \infty$, contradicting that $f(x) \le$ const.



Similarly, if there were a point q where $f'(q) \le 0$, then the graph of f lies above the tangent line at q. Consequently as $x \to -\infty$ then $f(x) \to \infty$, again contradicting the boundedness of f.

Remark: Note that the assumption $f''(x) \ge 0$ does not imply that $f'(x)$ is positive somewhere. An example is e^{-x}.

Part C 5 traditional problems (12 points each, so total is 60 points)

C–1. Let a_n be a sequence of complex numbers with $a_n \to A$ as $n \to \infty$ and let

$$S_n = \frac{a_1 + a_2 + \cdots + a_n}{n}.$$

Show that $S_n \to A$ as $n \to \infty$.

SOLUTION

$$S_n - A = \frac{a_1 + a_2 + \cdots + a_n}{n} - A = \frac{(a_1 - A) + (a_2 - A) + \cdots + (a_n - A)}{n}.$$

We reduce to the special case where $A = 0$ by letting $b_n = a_n - A$. Then $b_n \to 0$ and we need to show that

$$T_n := \frac{b_1 + b_2 + \cdots + b_n}{n} \to 0. \qquad (1)$$

Since $b_n \to 0$, Given $\epsilon > 0$ there is an N so that if $n > N$ then $|b_n| < \epsilon$. Rewrite (1) as the sum of two terms:

$$T_n = \frac{b_1 + b_2 + \cdots + b_N}{n} + \frac{b_{N+1} + b_{N+2} + \cdots + b_n}{n} = I_n + J_n.$$

By our choice of N, $|J_n| \le (n-N)\epsilon/n < \epsilon$. It is important to note that this will remain true if we choose an even larger value of n.

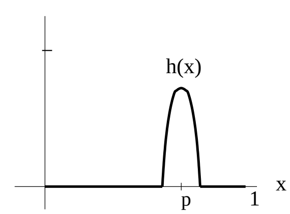
To estimate I_n, we use that since the sequence b_n converges, it is bounded. Thus for some M we know that $|b_n| < M$. Consequently, $|I_n| \le NM/n$. Now choose n so large that $NM/n < \epsilon$. Then

$$|T_n| \le |I_n| + |J_n| \; < \; 2\epsilon.$$

C–2. Let $f(x) \in C([0,1])$ have the property that $\int_0^1 f(x)h(x)\,dx = 0$ for it all functions $h \in C([0,1])$ with the additional property that $h(0) = h(1) = 0$. Prove that $f(x) \equiv 0$ on all of [0, 1].

SOLUTION By contradiction, say that $f(p) > 0$ at some $p \in [0,1]$. Because f is continuous, it will be positive at all nearby points. Thus we may assume that p is an interior point and also that $f > 0$ in a small interval J containing p. Let $h(x)$ be a continuous "bump" function with $h(x) > 0$ in J, and $h(x) = 0$ in the remainder of $[0,1]$ Then

$$\int_0^1 f(x)h(x)\,dx = \int_J f(x)h(x)\,dx > 0,$$



which is a contradiction.
The identical construction works if $f(p) < 0$ at some $p \in [0,1]$.

C–3. Let $\mathcal{M}$ be a metric space and let $B(\mathcal{M}; \mathbb{R})$ be the set of all bounded real-valued functions on $\mathcal{M}$ with the uniform norm:

$$\|f\| = \sup_{x\in\mathcal{M}} |f(x)|.$$

Since f is assumed to be bounded, then $\|f\| < \infty$. Define the distance between f and g to be $\|f - g\|$. This makes B into a metric space. Show that this is a *complete* metric space.

There are two steps: (i). Get a candidate for the limit function $f(x)$, and (ii). Prove that this $f(x)$ is bounded. [Where does your proof use that $\mathbb{R}$ is complete?]

Solution Let $f_n(x)$ be a sequence of bounded function that is a Cauchy sequence in this norm. In particular, at any point $x = p \in \mathcal{M}$, $f_n(p)$ is a Cauchy sequence of real numbers. Because of the completeness of the real numbers, the $f_n(p)$ converge of some real number, say q so we will define $f(p)$ to be this number q. This gives us the function to which the Cauchy sequence f_n converges.

We now need only show that $f(x)$ is a bounded function. This follows from the fact that a Cauchy sequence is always bounded. In fact, letting $\epsilon = 1$, there is an N_1 so that if $k > N_1$ then $\|f_k - f_{N_1}\| < 1$. Thus, for any point p

$$\begin{aligned} |f(p)| \leq& |f(p) - f_k(p)| + \|f_k - f_{N_1}\| + \|f_{N_1}\| \\ <& |f(p) - f_k(p)| + 1 + \|f_{N_1}\|. \end{aligned}$$

Since $f_k(p) \to f(p)$ the first term on the right can be made less than, say, 1, by choosing k large, while the last term in bounded because the functions we were working with were assumed to be bounded.

Consequently, the space of bounded continuous functions if complete.

C–4. Suppose that $G : \mathbb{R}^n \to \mathbb{R}^n$ is a continuous function with the property that for some real M

$$\|G(x) - G(y)\| \leq M\|x - y\| \qquad \text{for all } x, y \in \mathbb{R}^n. \tag{2}$$

Here $\|x\|$ is the standard Euclidean distance in $\mathbb{R}^n$.

If $\lambda > 0$ is small enough, show that the function $F : \mathbb{R}^n \to \mathbb{R}^n$ defined by

$$F(x) = x - \lambda G(x)$$

is one-to-one and onto, so for every $z \in \mathbb{R}^n$ the equation $F(x) = z$ has one and only one solution $x \in \mathbb{R}^n$. Note that a solution x is a fixed point of some map.

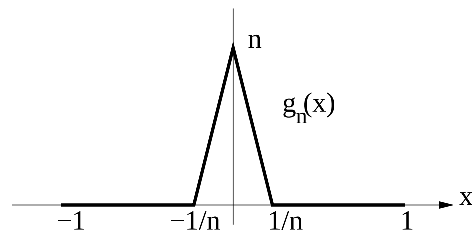
Solution We use a contracting mapping. For our complete metric space we use $\mathbb{R}^n$ with the Euclidean norm. The equation we want to solve is $x - \lambda G(x) = z$. We seek a fixed point of the map

$$T(x) := \lambda G(x) + z.$$

Clearly T maps $\mathbb{R}^n$ into itself so we only need to verify the contracting condition using the special property (2) of G:

$$\|T(x) - T(y)\| = \lambda\|G(x) - G(y)\| \le \lambda M\|x - y\|.$$

It is now clear that picking λ so that $\lambda M < 1$ the contracting assumption is satisfied. Thus,as desired, T has a unique fixed point. [REMARK: This problem is the essence of the Inverse Function Theorem].



C–5. Let $f(x) : \mathbb{R} \to \mathbb{R}$ be continuous for all x with $f(x) = 0$ for $|x| \ge 1$ and let $g_n(x)$ be the sequence of functions in the figure. Let

$$h_n(t) = \int_{-1}^{1} f(t - x)g_n(x)\,dx$$

a) Show that h_n is uniformly continuous.

SOLUTION

$$h_n(t) - h_n(s) = \int_{-1}^{1} [f(t - x) - f(s - x)]g_n(x)\,dx$$

Because $f(x)$ is continuous on $\mathbb{R}$ and zero for $|x| \ge 1$, it is uniformly continuous on $\mathbb{R}$. Given any $\epsilon > 0$ there is a δ so that if $|t - s| < \delta$ then $|f(t) - f(s)| < \epsilon$. But $(t - x) - (s - x) = t - s$ so $|f(t - x) - f(s - x)| < \epsilon$. Therefore,

$$|h_n(t) - h_n(s)| \le \int_{-1}^{1} \epsilon g_n(x)\,dx = \epsilon.$$

This shows that h_n is uniformly continuous.

b) Show that $\lim_{n\to\infty} h_n(t) = f(t)$ uniformly.

SOLUTION We use $f(t) = \int_{-1}^{1} f(t)g_n(x)\,dx$. Then

$$h_n(t) - f(t) = \int_{-1/n}^{1/n} [f(t - x) - f(t)]g_n(x)\,dx.$$

With δ from part (a), pick n so that $1/n < \delta$. Then $|(t - x) - t| = |x| < 1/n < \delta$ so that $|f(t - x) - f(t)| < \epsilon$. Therefore

$$|h_n(t) - f(t)| \le \int_{-1/n}^{1/n} \epsilon g_n(x)\,dx = \epsilon.$$

Because the right side is independent of t we have proved that h_n converges to f uniformly.