Dirichlet’s Principle

By 1840 it was known that if $S \subset \mathbb{R}$ is a closed and bounded set and $f : S \to \mathbb{R}$ is a continuous function, then there are points $p$ and $q$ in $S$ where $f$ has its maximum and minimum value.

Mathematicians and physicists were considering more complicated functions, such as, on a smooth surface $S$ in $\mathbb{R}^3$ finding the shortest path in the surface joining the two points $p$ and $q$. If we write the curve as $\vec{\gamma}(t) = (x(t), y(t), z(t)) \subset S$ where $\gamma(0) = p$ and $\gamma(1) = q$, then the length of the curve is

$$J(\vec{\gamma}) = \int_0^1 |\vec{\gamma}'(t)| \, dt.$$ 

To find the curve minimizing the distance we need to look at all curves in the surface and find the curve minimizing $J$. Thus we seek functions $x(t)$, $y(t)$, and $z(t)$. If the surface is smooth, is there always a minimizing curve? If so, is it smooth?

Historically, the first interesting problem of this sort was to study a bead, starting from rest, sliding down a curve under the influence of gravity. In particular, given the points $P$ and $Q$, find the equation of curve $y = f(x)$ from $P$ to $Q$ so the particle arrives at $Q$ in the least time. This is called the Brachistochrone Problem. The solution is interesting – and not at all obvious. Look it up.

One problem for a function $u(x, y)$ in several variables arose in a number of applications. Let $\Omega \subset \mathbb{R}^2$ be a bounded region with a smooth boundary $\partial \Omega$, and let $f(x, y)$ be a smooth function defined on the boundary, $\partial \Omega$. We seek a function $u(x, y)$ that minimizes the “energy”

$$J(v) = \iint_{\Omega} \left[ v_x^2 + v_y^2 \right] \, dx \, dy = \iint_{\Omega} |\nabla v|^2 \, dx \, dy \quad (1)$$

among all functions $v(x, y)$ that agree with $f$ on the boundary: $v(x, y) = f(x, y)$ for $(x, y) \in \partial \Omega$. In 1851, for his proof of what we call the Riemann Mapping Theorem, Riemann was seeking a minimizer since this minimizer it would be a solution of the Laplace equation:

$$u_{xx} + u_{yy} = 0 \quad \text{in } \Omega \quad \text{with } u = f \text{ on } \partial \Omega. \quad (2)$$

It is easy to show this. Since we want to minimize something, the idea is to use that at a minimum of a real-valued function $\varphi(t)$ its first derivative is zero.

The computation we will use to seek a minimum of the (smooth) function $J(v)$ closely follows that used for a real-valued function $f(X)$ of several variables, $X = (x_1, x_2, \ldots, x_n)$. We recall that. Say $f$ has a local minimum at an interior point $X_0$ of the set where $f$ is defined. For any vector $Z \in \mathbb{R}^n$ let

$$\varphi(t) = f(X_0 + tZ).$$

Observe that $f(X_0 + tZ) \geq f(X_0) = \varphi(0)$, that is, $\varphi(0) \leq \varphi(t)$ for all $t$ near zero. Thus $\varphi$ has a local min at $t = 0$ so its derivative at 0 is zero: $\varphi'(0) = 0$ (this is the directional derivative of $f$ at $X_0$ in the direction of the vector $Z$). But by the chain rule,

$$0 = \varphi'(0) = \nabla f(X_0) \cdot Z.$$
Because $Z$ is an arbitrary vector, this says that $\nabla f$ is orthogonal to all vectors so it must be zero, that is, $\nabla (f(X_0)) = 0$.

We use the same procedure to find the minima of $J(v)$ in equation (1). Say a function $u$ satisfying the boundary condition minimizes $J$. Let $h(x, y)$ be any smooth function that is zero on the boundary $\partial \Omega$. Then for any real $t$ the function $u(x, y) + th(x, y)$ also satisfies the boundary conditions. Thus the function

$$
\varphi(t) := J(u + th) = \iint_{\Omega} \left[ |\nabla u|^2 + 2t\nabla u \cdot \nabla h + t^2|\nabla h|^2 \right] \, dx \, dy
$$

has a minimum at $t = 0$. Therefore $\varphi'(0) = 0$. That is,

$$
\iint_{\Omega} \nabla u \cdot \nabla h \, dx \, dy = 0 \tag{3}
$$

for any smooth function $h$ that is zero on the boundary. Assuming the minimizer $u$ is smooth,

$$
\nabla u \cdot \nabla h = \nabla \cdot (h \nabla u) - h \Delta u,
$$

where $\Delta u = \nabla \cdot \nabla u = u_{xx} + u_{yy}$ is the Laplacian. Thus integrating by parts (the Divergence Theorem), equation (3) implies that

$$
\iint_{\Omega} (\Delta u)h \, dx \, dy = 0 \tag{4}
$$

for all $h$ that are zero on $\partial \Omega$. This implies that $\Delta u = 0$ throughout $\Omega$ (Proof: say $\Delta u > 0$ in a small disk $Q \subset \Omega$. Pick any $h$ that is positive on this disk and zero outside it. But then for this $h$ we have

$$
\iint_{\Omega} (\Delta u)h \, dx \, dy = \iint_{Q} (\Delta u)h \, dx \, dy > 0.
$$

contradicting equation (4). Thus, finding a minimizer of (1) gives a solution of the Laplace equation (2) with the desired boundary values.

Riemann’s innovation was using the existence of a minimizer of (1) to prove the existence of a solution of the boundary value problem (2). He call this Dirichlet’s Principle. Since in (1) $J(v)$ is bounded below (by zero), it is clear that $J$ has an infimum among all functions $v$ satisfying the boundary condition. It is not at all clear that there is a twice differentiable function $u$ that actually minimizes $J$. To illustrate the difficulty Weierstrass gave an explicit example of a related problem

Minimize $J(v) := \int_{-1}^{1} x^2 v'^2(x) \, dx$

for all $v$ with $v(-1) = -1$ and $v(1) = 1$. Following his reasoning, we show that $J$ have an infimum but does not have a minimum. He considered the sequence of functions

$$
v_n(x) = \begin{cases} 
-1 & \text{if } -1 \leq x \leq -1/n, \\
 \frac{nx}{n} & \text{if } -1/n \leq x \leq 1/n, \\
 1 & \text{if } 1/n \leq x \leq 1
\end{cases}
$$
By an easy calculation has \( J(v_n) = 2/(3n) \to 0 \). This shows that the \( \inf J(v) = 0 \) (if you prefer a smooth sequence of functions you can use \( v_n(x) := \frac{\tanh nx}{\tanh n} \)). However, if there is a \( v \) with \( J(v) = 0 \) then \( v' = 0 \), so \( v \) must be a constant– and that can’t satisfy the boundary conditions. Thus this \( J \) has an \( \inf \) but not a \( \min \).

Mathematicians generally believed the idea behind Riemann’s proof of the existence of a solution to (2) – but there certainly was a gap in the proof. It took about 50 years to develop the ideas such as compactness needed to understand the situation adequately.

**Toy Example:** Here is a toy (but not obvious) example where the idea behind Dirichlet’s Principle works immediately. Say you are seeking a solution \((x, y)\) of the two equations

\[
\begin{align*}
2x(x^2 + y^2) + y - 1 &= 0 \\
2y(x^2 + y^2) + 2y^3 + x + 2 &= 0
\end{align*}
\]

(5)

Idea: find a function \( f(x, y) \) that has a local minimum somewhere and with the property that equations (5) are the equations \( f_x = 0 \) and \( f_y = 0 \), so they will be satisfied at this local minimum.

Consider the function

\[
 f(x, y) = (x^2 + y^2)^2 + y^4 + 2xy - 2x - 4y - 3
\]

Computing \( f_x \) and \( f_y \), except for a factor of 2, these are exactly the equations (5) we wanted to solve. Thus, if we can show that \( f \) has a local minimum somewhere, then at least one solution exists, namely \((x_0, y_0)\).

With this problem in mind, in Homework Set 2 Problem 4 you found a number \( R \) so that if \( x^2 + y^2 \geq R^2 \) then \( f(x, y) \geq 1 \). Since the disk \( x^2 + y^2 \leq R^2 \) is compact, there is at least one point \((x_0, y_0)\) in this disk where \( f \) attains its minimum. Because \( f(0, 0) = -3 < 1 \), this point is not on the boundary of the disk so it is an interior point. Thus, at this point, the gradient of \( f \) is zero, that is, \( f_x(x_0, y_0) = 0 \) and \( f_y(x_0, y_0) = 0 \).

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