1.39. **Definition.** *Field Axioms.* A set $S$ with operations $+$ and $\cdot$ and distinguished elements $0$ and $1$ with $0 \neq 1$ is a **field** if the following properties hold for all $x, y, z \in S$.

A0: $x + y \in S$
A1: $(x+y)+z = x+(y+z)$
A2: $x + y = y + x$
A3: $x + 0 = x$
A4: given $x$, there is a $w \in S$ such that $x + w = 0$

M0: $x \cdot y \in S$
M1: $(x \cdot y) \cdot z = x \cdot (y \cdot z)$
M2: $x \cdot y = y \cdot x$
M3: $x \cdot 1 = x$
M4: for $x \neq 0$, there is a $w \in S$ such that $x \cdot w = 1$

**Closure**

**Associativity**

**Commutativity**

**Identity**

**Inverse**

**Distributive Law**

The operations $+$ and $\cdot$ are called **addition** and **multiplication**. The elements $0$ and $1$ are the **additive identity element** and the **multiplicative identity element**.
It follows from these axioms that the additive inverse and multiplicative inverse (of a nonzero \( x \)) are unique. The additive inverse of \( x \) is the \textbf{negative} of \( x \), written as \(-x\). To define \textbf{subtraction} of \( y \) from \( x \), we let \( x - y = x + (-y) \). The multiplicative inverse of \( x \) is the \textbf{reciprocal} of \( x \), written as \( x^{-1} \). The element 0 has no reciprocal. To define \textbf{division} of \( x \) by \( y \) when \( y \neq 0 \), we let \( x/y = x \cdot (y^{-1}) \). We write \( x \cdot y \) as \( xy \) and \( x \cdot x \) as \( x^2 \). We use parentheses where helpful to clarify the order of operations.
1.40. Definition. *Order Axioms.* A positive set in a field $F$ is a set $P \subseteq F$ such that for $x, y \in F$,

P1: $x, y \in P$ implies $x + y \in P$ \text{ Closure under Addition}

P2: $x, y \in P$ implies $xy \in P$ \text{ Closure under Multiplication}

P3: $x \in F$ implies exactly one of $x = 0, x \in P, -x \in P$ \text{ Trichotomy}

An ordered field is a field with a positive set $P$. In an ordered field, we define $x < y$ to mean $y - x \in P$. The relations $\leq, <, \text{ and } \geq$ have analogous definitions in terms of $P$. 
1.43. Proposition. Elementary consequences of the field axioms.

a) \( x + z = y + z \) implies \( x = y \)

b) \( x \cdot 0 = 0 \)

c) \( (-x)y = -(xy) \)

d) \( -x = (-1)x \)

e) \( (-x)(-y) = xy \)

f) \( xz = yz \) and \( z \neq 0 \) imply \( x = y \)

g) \( xy = 0 \) implies \( x = 0 \) or \( y = 0 \)
1.44. Proposition. Properties of an ordered field.

O1: $x \leq x$  Reflexive Property
O2: $x \leq y$ and $y \leq x$ imply $x = y$  Antisymmetric Property
O3: $x \leq y$ and $y \leq z$ imply $x \leq z$  Transitive Property
O4: at least one of $x \leq y$ and $y \leq x$ holds  Total Ordering Property
1.45. Proposition. More properties of an ordered field.

F1: $x \leq y$ implies $x + z \leq y + z$  
Additive Order Law

F2: $x \leq y$ and $0 \leq z$ imply $xz \leq yz$  
Multiplicative Order Law

F3: $x \leq y$ and $u \leq v$ imply $x + u \leq y + v$  
Addition of Inequalities

F4: $0 \leq x \leq y$ and $0 \leq u \leq v$ imply $xu < yv$  
Multiplication of Inequalities
1.46. Proposition. Still more properties of an ordered field.

a) $x \leq y$ implies $-y \leq -x$

b) $x \leq y$ and $z \leq 0$ imply $yz \leq xz$

c) $0 \leq x$ and $0 \leq y$ imply $0 \leq xy$

d) $0 \leq x^2$

e) $0 < 1$

f) $0 < x$ implies $0 < x^{-1}$

g) $0 < x < y$ implies $0 < y^{-1} < x^{-1}$