1.39. Definition. *Field Axioms.* A set S with operations $+$ and $\cdot$ and distinguished elements 0 and 1 with $0 \neq 1$ is a **field** if the following properties hold for all $x, y, z \in S$.

A0: $x + y \in S$	M0: $x \cdot y \in S$	Closure
A1: $(x+y)+z = x+(y+z)$	M1: $(x \cdot y) \cdot z = x \cdot (y \cdot z)$	Associativity
A2: $x + y = y + x$	M2: $x \cdot y = y \cdot x$	Commutativity
A3: $x + 0 = x$	M3: $x \cdot 1 = x$	Identity
A4: given x, there is a $w \in S$ such that $x + w = 0$	M4: for $x \neq 0$, there is a $w \in S$ such that $x \cdot w = 1$	Inverse
	DL: $x \cdot (y + z) = x \cdot y + x \cdot z$	Distributive Law

The operations $+$ and $\cdot$ are called **addition** and **multiplication**. The elements 0 and 1 are the **additive identity element** and the **multiplicative identity element**.

It follows from these axioms that the additive inverse and multiplicative inverse (of a nonzero x) are unique. The additive inverse of x is the **negative** of x, written as $-x$. To define **subtraction** of y from x, we let $x - y = x + (-y)$. The multiplicative inverse of x is the **reciprocal** of x, written as x^{-1}. The element 0 has no reciprocal. To define **division** of x by y when $y \neq 0$, we let $x/y = x \cdot (y^{-1})$. We write $x \cdot y$ as xy and $x \cdot x$ as x^2. We use parentheses where helpful to clarify the order of operations.

1.40. Definition. *Order Axioms.* A **positive set** in a field F is a set $P \subseteq F$ such that for $x, y \in F$,

P1: $x, y \in P$ implies $x + y \in P$	Closure under Addition
P2: $x, y \in P$ implies $xy \in P$	Closure under Multiplication
P3: $x \in F$ implies exactly one of $x = 0,\ x \in P,\ -x \in P$	Trichotomy

An **ordered field** is a field with a positive set P. In an ordered field, we define $x < y$ to mean $y - x \in P$. The relations $\leq$, $<$, and $\geq$ have analogous definitions in terms of P.

1.43. Proposition. *Elementary consequences of the field axioms.*

a) $x + z = y + z$ implies $x = y$

b) $x \cdot 0 = 0$

c) $(-x)y = -(xy)$

d) $-x = (-1)x$

e) $(-x)(-y) = xy$

f) $xz = yz$ and $z \neq 0$ imply $x = y$

g) $xy = 0$ implies $x = 0$ or $y = 0$

1.44. Proposition. *Properties of an ordered field.*

O1: $x \le x$	Reflexive Property
O2: $x \le y$ and $y \le x$ imply $x = y$	Antisymmetric Property
O3: $x \le y$ and $y \le z$ imply $x \le z$	Transitive Property
O4: at least one of $x \le y$ and $y \le x$ holds	Total Ordering Property

1.45. Proposition. *More properties of an ordered field.*

F1: $x \leq y$ implies $x + z \leq y + z$	Additive Order Law
F2: $x \leq y$ and $0 \leq z$ imply $xz \leq yz$	Multiplicative Order Law
F3: $x \leq y$ and $u \leq v$ imply $x + u \leq y + v$	Addition of Inequalities
F4: $0 \leq x \leq y$ and $0 \leq u \leq v$ imply $xu < yv$	Multiplication of Inequalities

1.46. Proposition. *Still more properties of an ordered field.*

a) $x \leq y$ implies $-y \leq -x$

b) $x \leq y$ and $z \leq 0$ imply $yz \leq xz$

c) $0 \leq x$ and $0 \leq y$ imply $0 \leq xy$

d) $0 \leq x^2$

e) $0 < 1$

f) $0 < x$ implies $0 < x^{-1}$

g) $0 < x < y$ implies $0 < y^{-1} < x^{-1}$