7.5.3 Theorem *Let $f : [-a,a] \times B(x_0,r) \to \mathbb{R}^n$ be a continuous mapping. Let $C = \sup\{\|f(t,x)\| \mid (t,x) \in [-a,a] \times B(x_0,r)\}$. Suppose there is a constant K such that*

$$\|f(t,x) - f(t,y)\| \leq K\|x - y\| \tag{L}$$

for all $t \in [-a,a]$, $x, y \in B(x_0,r)$. Let $b < \min\{a, r/C, 1/K\}$. Then there is a unique continuously differentiable map $x : [-b,b] \to B(x_0,r) \subset \mathbb{R}^n$ such that

$$x(0) = x_0 \quad (\textit{initial condition}) \quad \textit{and} \quad \frac{dx}{dt} = f(t, x(t)).$$

Proof The differential equation and the initial condition $x(0) = x_0$ are equivalent, by the fundamental theorem of calculus, to the condition

$$x(t) = x_0 + \int_0^t f(s, x(s))\, ds$$

for a continuous function $x(t)$. Consider $C([-b,b], \mathbb{R}^n)$, which we know (from §5.4) is a complete metric space. Let

$$A = \{\varphi \in C([-b,b], \mathbb{R}^n) \mid \varphi(0) = x_0 \text{ and } \varphi(t) \in B(x_0,r)\}.$$

Then $A \subset C([-b,b], \mathbb{R}^n)$ is closed (why?), and therefore A is also a complete metric space. We will apply the contraction mapping principle to this space A. Define $F : A \to A$ by

$$F(\varphi)(t) = x_0 + \int_0^t f(s, \varphi(s))\, ds.$$

Here, the expression $\int_0^t f(s,\varphi(s))\, ds$ is obtained by integrating each component of f; the result is a vector. The general inequality

$$\left\| \int_0^t f(s, \varphi(s))\, ds \right\| \leq \int_0^t \|f(s, \varphi(s))\|\, ds$$

is analogous to the similar result for the case of real functions, which we accept here (see §4.8). First, we must show that $F(\varphi) \in A$. The function $F(\varphi)$ is continuous, since

$$\|F(\varphi)(\tau) - F(\varphi)(t)\| = \left\| \int_\tau^t f(s, \varphi(s))\, ds \right\| \leq \int_t^\tau \|f(s, \varphi(s))\|\, ds \leq C|\tau - t|$$

for $-b \leq t < \tau \leq b$. Thus, $F(\varphi) \in C([-b,b], \mathbb{R}^n)$. Also, $F(\varphi)(0) = x_0$, and for all $t \in [-b,b]$,

$$\|F(\varphi)(t) - x_0\| = \left\| \int_0^t f(s, \varphi(s))\, ds \right\| \leq \operatorname{sign}(t) \int_0^t \|f(s, \varphi(s))\, ds\| \leq b \cdot C < r,$$

7.5.3 Theorem *Let $f : [-a, a] \times B(x_0, r) \to \mathbb{R}^n$ be a continuous mapping. Let $C = \sup\{\|f(t,x)\| \mid (t,x) \in [-a,a] \times B(x_0,r)\}$. Suppose there is a constant K such that*

$$\|f(t,x) - f(t,y)\| \le K\|x - y\| \qquad \text{(L)}$$

for all $t \in [-a, a]$, $x, y \in B(x_0, r)$. Let $b < \min\{a, r/C, 1/K\}$. Then there is a unique continuously differentiable map $x : [-b, b] \to B(x_0, r) \subset \mathbb{R}^n$ such that

$$x(0) = x_0 \ \ (\textit{initial condition}) \ \ \textit{and} \ \ \frac{dx}{dt} = f(t, x(t)).$$

Proof The differential equation and the initial condition $x(0) = x_0$ are equivalent, by the fundamental theorem of calculus, to the condition

$$x(t) = x_0 + \int_0^t f(s, x(s))\, ds$$

for a continuous function $x(t)$. Consider $C([-b, b], \mathbb{R}^n)$, which we know (from §5.4) is a complete metric space. Let

$$A = \{\varphi \in C([-b, b], \mathbb{R}^n) \mid \varphi(0) = x_0 \text{ and } \varphi(t) \in B(x_0, r)\}.$$

Then $A \subset C([-b, b], \mathbb{R}^n)$ is closed (why?), and therefore A is also a complete metric space. We will apply the contraction mapping principle to this space A. Define $F : A \to A$ by

$$F(\varphi)(t) = x_0 + \int_0^t f(s, \varphi(s))\, ds.$$

Here, the expression $\int_0^t f(s, \varphi(s))\, ds$ is obtained by integrating each component of f; the result is a vector. The general inequality

$$\left\| \int_0^t f(s, \varphi(s))\, ds \right\| \le \int_0^t \|f(s, \varphi(s))\|\, ds$$

is analogous to the similar result for the case of real functions, which we accept here (see §4.8). First, we must show that $F(\varphi) \in A$. The function $F(\varphi)$ is continuous, since

$$\|F(\varphi)(\tau) - F(\varphi)(t)\| = \left\| \int_\tau^t f(s, \varphi(s))\, ds \right\| \le \int_t^\tau \|f(s, \varphi(s))\|\, ds \le C|\tau - t|$$

for $-b \le t < \tau \le b$. Thus, $F(\varphi) \in C([-b, b], \mathbb{R}^n)$. Also, $F(\varphi)(0) = x_0$, and for all $t \in [-b, b]$,

$$\|F(\varphi)(t) - x_0\| = \left\| \int_0^t f(s, \varphi(s))\, ds \right\| \le \operatorname{sign}(t) \int_0^t \|f(s, \varphi(s))\, ds\| \le b \cdot C < r,$$

since $b < r/C$. The factor sign (t) takes into account the possibility that $t < 0$. Thus $F(\varphi)(t) \in B(x_0, r)$, and so $F(\varphi) \in A$. Next, for $\varphi, \psi \in A$,

$$\begin{aligned} \|F(\varphi) - F(\psi)\| &= \sup_{-b \le t \le b} \|F(\varphi)(t) - F(\psi)(t)\| \\ &= \sup_{-b \le t \le b} \left\| \int_0^t f(s, \varphi(s)) - f(s, \psi(s))\, ds \right\| \\ &\le \sup_{-b \le t \le b} \operatorname{sign}(t) \int_0^t \|f(s, \varphi(s)) - f(s, \psi(s))\|\, ds \\ &\le \sup_{-b \le t \le b} \operatorname{sign}(t) \int_0^t K\|\varphi(s) - \psi(s)\|\, ds \\ &\le \sup_{-b \le t \le b} K \operatorname{sign}(t) \int_0^t \|\varphi - \psi\|\, ds \le Kb\|\varphi - \psi\| \end{aligned}$$

where $Kb < 1$. Therefore, if we let $k = b \cdot K < 1$, then $d(F(\varphi), F(\psi)) \le kd(\varphi, \psi)$, and so F is a contraction and thus has a unique fixed point: $x = F(x)$. This fixed point $x(t)$ is the unique solution we were seeking. ■

The iteration scheme mentioned in the text comes about because, as we saw in the proof of the contraction mapping theorem, the unique fixed point is the limit $F^n(\varphi)$ as $n \to \infty$ for any $\varphi \in A$, such as $\varphi(t) \equiv x_0$.

7.6.1 Morse Lemma *Let $A \subset \mathbb{R}^n$ be open and $f : A \to \mathbb{R}$ a smooth (infinitely differentiable) function. Suppose $\mathbf{D}f(x_0) = 0$ and the Hessian of f at x_0 is nonsingular. Then there are a neighborhood U of x_0 and a neighborhood V of 0 in $\mathbb{R}^n$ and a smooth map $g : V \to U$ with a smooth inverse such that $f \circ g = h$ has the form*

$$h(y) = f(x_0) - [y_1^2 + y_2^2 + \cdots + y_\lambda^2] + [y_{\lambda+1}^2 + \cdots + y_n^2],$$

where λ is some fixed integer between 0 and n.

Proof We lose no generality if we assume $x_0 = 0$ and $f(x_0) = 0$. Write

$$f(x_1, \ldots, x_n) = \int_0^1 \frac{df(tx_1, \ldots, tx_n)}{dt}\, dt = \int_0^1 \sum_{i=1}^n x_i \frac{\partial f}{\partial x_i}(tx_1, \ldots, tx_n)\, dt.$$

If we set

$$g_i(x_1, \ldots, x_n) = \int_0^1 \frac{\partial f}{\partial x_i}(tx_1, \ldots, tx_n)\, dt,$$