7.5.3 Theorem  Let \( f : [-a, a] \times B(x_0, r) \rightarrow \mathbb{R}^n \) be a continuous mapping. Let 
\[ C = \sup \{ ||f(t,x)|| \mid (t,x) \in [-a,a] \times B(x_0, r) \}. \]
Suppose there is a constant \( K \) such that
\[ ||f(t,x) - f(t,y)|| \leq K ||x - y|| \]  
for all \( t \in [-a,a], x,y \in B(x_0, r). \) Let \( b < \min\{a, r/C, 1/K\}. \) Then there is a unique continuously differentiable map \( x : [-b,b] \rightarrow B(x_0, r) \subset \mathbb{R}^n \) such that
\[ x(0) = x_0 \quad \text{(initial condition)} \quad \text{and} \quad \frac{dx}{dt} = f(t,x(t)). \]

Proof  The differential equation and the initial condition \( x(0) = x_0 \) are equivalent, by the fundamental theorem of calculus, to the condition
\[ x(t) = x_0 + \int_0^t f(s,x(s)) \, ds \]
for a continuous function \( x(t) \). Consider \( C([-b,b], \mathbb{R}^n) \), which we know (from §5.4) is a complete metric space. Let
\[ A = \{ \varphi \in C([-b,b], \mathbb{R}^n) \mid \varphi(0) = x_0 \text{ and } \varphi(t) \in B(x_0, r) \}. \]
Then \( A \subset C([-b,b], \mathbb{R}^n) \) is closed (why?), and therefore \( A \) is also a complete metric space. We will apply the contraction mapping principle to this space \( A \). Define \( F : A \rightarrow A \) by
\[ F(\varphi)(t) = x_0 + \int_0^t f(s, \varphi(s)) \, ds. \]
Here, the expression \( \int_0^t f(s, \varphi(s)) \, ds \) is obtained by integrating each component of \( f \); the result is a vector. The general inequality
\[ \left\| \int_0^t f(s, \varphi(s)) \, ds \right\| \leq \int_0^t ||f(s, \varphi(s))|| \, ds \]
is analogous to the similar result for the case of real functions, which we accept here (see §4.8). First, we must show that \( F(\varphi) \in A \). The function \( F(\varphi) \) is continuous, since
\[ ||F(\varphi)(\tau) - F(\varphi)(t)|| = \left\| \int_t^\tau f(s, \varphi(s)) \, ds \right\| \leq \int_t^\tau ||f(s, \varphi(s))|| \, ds \leq C|\tau - t| \]
for \(-b \leq t < \tau \leq b\). Thus, \( F(\varphi) \in C([-b,b], \mathbb{R}^n) \). Also, \( F(\varphi)(0) = x_0 \), and for all \( t \in [-b,b] \),
\[ ||F(\varphi)(t) - x_0|| = \left\| \int_0^t f(s, \varphi(s)) \, ds \right\| \leq \text{sign}(t) \int_0^t ||f(s, \varphi(s))|| \, ds \leq b \cdot C < r, \]
7.5.3 Theorem Let \( f : [-a, a] \times B(x_0, r) \to \mathbb{R}^n \) be a continuous mapping. Let \( C = \sup \{\|f(t,x)\| \mid (t,x) \in [-a, a] \times B(x_0, r)\} \). Suppose there is a constant \( K \) such that

\[
\|f(t,x) - f(t,y)\| \leq K\|x - y\|
\]

for all \( t \in [-a, a], \ x, y \in B(x_0, r) \). Let \( b < \min\{a, r/C, 1/K\} \). Then there is a unique continuously differentiable map \( x : [-b, b] \to B(x_0, r) \subset \mathbb{R}^n \) such that

\[
x(0) = x_0 \quad \text{(initial condition)} \quad \text{and} \quad \frac{dx}{dt} = f(t, x(t)).
\]

Proof The differential equation and the initial condition \( x(0) = x_0 \) are equivalent, by the fundamental theorem of calculus, to the condition

\[
x(t) = x_0 + \int_0^t f(s, x(s)) \, ds
\]

for a continuous function \( x(t) \). Consider \( C([-b, b], \mathbb{R}^n) \), which we know (from §5.4) is a complete metric space. Let

\[
A = \{\varphi \in C([-b, b], \mathbb{R}^n) \mid \varphi(0) = x_0 \text{ and } \varphi(t) \in B(x_0, r)\}.
\]

Then \( A \subset C([-b, b], \mathbb{R}^n) \) is closed (why?), and therefore \( A \) is also a complete metric space. We will apply the contraction mapping principle to this space \( A \). Define \( F : A \to A \) by

\[
F(\varphi)(t) = x_0 + \int_0^t f(s, \varphi(s)) \, ds.
\]

Here, the expression \( \int_0^t f(s, \varphi(s)) \, ds \) is obtained by integrating each component of \( f \); the result is a vector. The general inequality

\[
\left\| \int_0^t f(s, \varphi(s)) \, ds \right\| \leq \int_0^t \| f(s, \varphi(s)) \| \, ds
\]

is analogous to the similar result for the case of real functions, which we accept here (see §4.8). First, we must show that \( F(\varphi) \in A \). The function \( F(\varphi) \) is continuous, since

\[
\|F(\varphi)(\tau) - F(\varphi)(t)\| = \left\| \int_t^\tau f(s, \varphi(s)) \, ds \right\| \leq \int_t^\tau \| f(s, \varphi(s)) \| \, ds \leq C|\tau - t|
\]

for \(-b \leq t < \tau \leq b\). Thus, \( F(\varphi) \in C([-b, b], \mathbb{R}^n) \). Also, \( F(\varphi)(0) = x_0 \), and for all \( t \in [-b, b] \),

\[
\|F(\varphi)(t) - x_0\| = \left\| \int_0^t f(s, \varphi(s)) \, ds \right\| \leq \text{sign}(t) \int_0^t \| f(s, \varphi(s)) \| \, ds \leq b \cdot C < r,
\]
Theorem Proofs for Chapter 7

since \( b < r/C \). The factor sign \( (t) \) takes into account the possibility that \( t < 0 \). Thus \( F(\varphi)(t) \in B(x_0, r) \), and so \( F(\varphi) \in A \). Next, for \( \varphi, \psi \in A \),

\[
\|F(\varphi) - F(\psi)\| = \sup_{-b \leq t \leq b} \|F(\varphi)(t) - F(\psi)(t)\|
\]

\[
= \sup_{-b \leq t \leq b} \left\| \int_0^t f(s, \varphi(s)) - f(s, \psi(s)) \, ds \right\|
\]

\[
\leq \sup_{-b \leq t \leq b} \text{sign}(t) \int_0^t \|f(s, \varphi(s)) - f(s, \psi(s))\| \, ds
\]

\[
\leq \sup_{-b \leq t \leq b} \text{sign}(t) \int_0^t K \|\varphi(s) - \psi(s)\| \, ds
\]

\[
\leq \sup_{-b \leq t \leq b} K \text{sign}(t) \int_0^t \|\varphi - \psi\| \, ds \leq K b \|\varphi - \psi\|
\]

where \( K b < 1 \). Therefore, if we let \( k = b \cdot K < 1 \), then \( d(F(\varphi), F(\psi)) \leq k d(\varphi, \psi) \), and so \( F \) is a contraction and thus has a unique fixed point: \( x = F(x) \). This fixed point \( x(t) \) is the unique solution we were seeking. \( \Box \)

The iteration scheme mentioned in the text comes about because, as we saw in the proof of the contraction mapping theorem, the unique fixed point is the limit \( F^n(\varphi) \) as \( n \to \infty \) for any \( \varphi \in A \), such as \( \varphi(t) \equiv x_0 \).

7.6.1 Morse Lemma  Let \( A \subseteq \mathbb{R}^n \) be open and \( f : A \to \mathbb{R} \) a smooth (infinitely differentiable) function. Suppose \( Df(x_0) = 0 \) and the Hessian of \( f \) at \( x_0 \) is nonsingular. Then there are a neighborhood \( U \) of \( x_0 \) and a neighborhood \( V \) of \( 0 \) in \( \mathbb{R}^n \) and a smooth map \( g : V \to U \) with a smooth inverse such that \( f \circ g = h \) has the form

\[
h(y) = f(x_0) - [y_1^2 + y_2^2 + \cdots + y_\lambda^2] + [y_{\lambda+1}^2 + \cdots + y_n^2],
\]

where \( \lambda \) is some fixed integer between 0 and \( n \).

Proof  We lose no generality if we assume \( x_0 = 0 \) and \( f(x_0) = 0 \). Write

\[
f(x_1, \ldots, x_n) = \int_0^1 \frac{df(tx_1, \ldots, tx_n)}{dt} \, dt = \int_0^1 \sum_{i=1}^n x_i \frac{\partial f}{\partial x_i}(tx_1, \ldots, tx_n) \, dt.
\]

If we set

\[
g_i(x_1, \ldots, x_n) = \int_0^1 \frac{\partial f}{\partial x_i}(tx_1, \ldots, tx_n) \, dt,
\]