Math 508, Fall 2014 Jerry L. Kazdan

Math 508, Notes for Sept. 4, 2014

Remarks from Tuesday
1. Proving $1 > 0$ using if $y > 0$, then $1y = y > 0$ so $1 > 0$.
This uses:

- $\diamond$ There *exists* some positive element y,
- $\diamond$ "If $y > 0$ and $xy > 0$ then $x > 0$".

Both are easy to prove – but so far we have not proved them. Note that we want to prove the equally basic statement $1 > 0$.

2. Intuition why $(-x)(-y) = xy$

Extended real numbers $-\infty < x < \infty$

Complex Numbers
Think of $z = a + ib$, where a and b are real as a point in the plane. Then we have $0 = (0, 0)$, $1 = (1, 0)$, and $i = (0, 1)$; a is the real and b the imaginary parts of z, $a = Re\{z\}$, $b = \Im\{z\}$.
Add them like vectors: If $w = (c, d)$, then $z + w = (a + c,\ b + d$.
Multiply them: $zw = (ac - bd,\ ad + bc$.
Claim: these form a field. The only complication is multiplicative inverses. Idea:

$$\frac{1}{z} = \frac{1}{a+ib} = \frac{1}{a+ib}\left(\frac{a-ib}{a-ib}\right) = \frac{a-ib}{a^2+b^2} = \frac{a}{a^2+b^2} - \frac{b}{a^2+b^2}i.$$

Introduce $\bar{z} = a - ib$. Then $z\bar{z} = a^2 + b^2$ so if $z \neq 0$

$$\frac{1}{z} = \frac{\bar{z}}{z\bar{z}}.$$

Note $z + \bar{z} = 2a = 2\mathrm{Re}\{z\}$ and $z - \bar{z} = 2b = 2\mathrm{Im}\{z\}$.

Absolute value $|z| = \sqrt{a^2 + b^2} = \sqrt{z\bar{z}}$

Facts:
AV-1: $|z| \geq 0$, equal zero iff $z = 0$.
AV-2: $|zw| = |z||w|$
AV-3: $|z + w| \leq |z| + |w|$ (triangle inequality)

Complex numbers of the form $z = a + 0i = (a, 0)$ are just the real numbers. Thus the real numbers from a *sub-field* of the complex numbers.

Exercise (Geometric series)). For any complex number $z \neq 1$:

$$1 + z + z^2 + \cdots + z^n = \frac{1 - z^{n+1}}{1 - z}.$$

In fact, the proof for the real numbers works without change for any field.

If $a \in \mathbb{C}$ and $r \geq 0 \in \mathbb{R}$ the set

$$B_a(r) = \{z \in \mathbb{C} : |z - a| < r\}$$

is the *disk* in the complex plane with center a and radius r while

$$S_a(r) = \{z \in \mathbb{C} : |z - a| = r\}$$

is the *circle* that is the boundary of this disk.

Remark Above we formally used the model of a complex number $z = a + ib$ as an ordered pair of real numbers (a, b) with a careful definition of addition and multiplication. Another useful model is as the set of 2×2 matrices of the form

$$Z = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$$

with the usual definition of addition and multiplication of matrices. Here the special matrix $J = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ has the property that $J^2 = -I$ and plays the role of the complex number i. In this notation, $Z = aI + bJ$.

Real Euclidean Space: $\mathbb{R}^n$

Points $X = (x_1, \ldots, x_n)$, $Y = (y_1, \ldots, y_n$, where the x_j and y_j are real numbers.

Algebra We use the usual addition and multiplication by scalars $c \in \mathbb{R}$.

$$X + Y = (x_1 + y_1,\, x_2 + y_2, \ldots, x_n + y_n) \qquad cX = (cx_1,\, cx_2, \ldots, cx_n).$$

The distributive rules hold for ant scalars a, b, c:

$$(a + b)X = aX + bX, \qquad c(X + Y) = cX + cY$$

Inner (or "dot") Product:

$$X \cdot Y = \langle X, Y\rangle = x_1y_1 + x_2y_2 + \cdots + x_ny_n.$$

Properties of an inner product:
IP-1. $\langle X, X\rangle \geq 0$. $\langle X, X\rangle = 0$ iff $X = 0$.
IP-2. $\langle Y, X\rangle = \langle X, Y\rangle$ (symmetry)
IP-3. $\langle cX, Y\rangle = c\langle Y, X\rangle$
IP-4. $\langle X + Y, Z\rangle = \langle X, Z\rangle + \langle Y, Z\rangle$.
These are just the axioms for an inner product on a real vector space (see below for the modification is the field of scalars is the complex numbers C).

Example

$$\begin{aligned} \langle X + Y, X + Y\rangle =& \langle X, X + Y\rangle + \langle Y, X + Y\rangle \\ =& \langle X, X\rangle + 2\langle X, Y\rangle + \langle Y, Y\rangle. \end{aligned} \tag{1}$$

NORM: $|X| = \sqrt{x_1^2 + \cdots + x_n^2}$.
Properties:
N-1. $|X| \geq 0$. $|X| = 0$ iff $X = 0$.
N-2. For any scalar c, $|cX| = |c||X|$.
N-3. $|X + Y| \leq |X| + |Y|$ (triangle inequality – proved below).

Note $|X|^2 = \langle X, X \rangle$. As general procedure, in any computation involving norms in a space with an inner product, square the norm and then use the algebraic properties of the inner product (see the Pythagorean example below).

GEOMETRIC INTERPRETATION OF THE INNER PRODUCT
While the inner product is easy to compute, it is not obvious how to interpret it – or if it is even useful. The key is that in the plane $\mathbb{R}^2$, then

$$\langle X, Y \rangle = |X||Y| \cos\theta, \tag{2}$$

where θ is the angle between X and Y. This formula is a consequence of the formula for $\cos(\theta + \phi)$. In this case we see that X and Y are it orthogonal (perpendicular: $X \perp Y$) precisely when $\cos\theta = 0$, that is, $\langle X, Y \rangle = 0$. This leads us to define (in any inner product space) that X and Y are orthogonal if $\langle X, Y \rangle = 0$.

EXAMPLE (Pythagoras) In a real inner product space,

$$|X + Y|^2 = |X|^2 + |Y|^2$$

if and only if X and Y are orthogonal.
Proof: This is obvious from the identity (1) above.

In the plane, the identity (2) implies the *Cauchy-Schwarz* inequality

$$|\langle X, Y \rangle| \leq |X||Y|. \tag{3}$$

We will now prove this inequality directly, only using the four properties of the inner product. To avoid trivialities, assume $Y \neq 0$. Note that for any scalar $t \in \mathbb{R}$

$$0 \leq |X - tY|^2 = |X|^2 - 2t\langle X, Y \rangle + t^2|Y|^2.$$

Pick t to minimize this, say by taking the derivative. We find $t = \langle X, Y \rangle / |Y|^2$. Using this value the above inequality gives

$$\begin{aligned} 0 \leq& |X|^2 - 2\langle X, Y \rangle^2/|Y|^2 + \langle X, Y \rangle^2/|Y|^2 \\ =& |X|^2 - \langle X, Y \rangle^2/|Y|^2, \end{aligned}$$

which is just what we wanted.

The triangle inequality is an immediate consequence of the Cauchy-Schwaz inequality since from equation (1)

$$\begin{aligned} |X+Y|^2 \leq& |X|^2 + 2|X||Y| + |Y|^2 \\ =& (|X|+|Y|)^2. \end{aligned}.$$

Example For continuous functions, say on the interval $-\pi < x < \pi$, we will often use the (standard) inner product

$$\langle f,\, g\rangle = \int_{-\pi}^{\pi} f(x)g(x)\, dx. \tag{4}$$

It satisfies all the four properties(see above) of an inner product so anything we prove using only these four properties will automatically be valid for this case. One example is that since for any integers k and ℓ

$$\int_{-\pi}^{\pi} \cos kx \sin \ell x \, dx = 0$$

then in this inner product the functions $\cos kx$ and $\sin \ell x$ are orthogonal. This is important in the study of Fourier series.

Since we only used the axioms for an inner product, our proof of the Cauchy-Schwarz inequality implies that

$$\left| \int_{-\pi}^{\pi} f(x)g(x)\, dx \right| \leq \left[\int_{-\pi}^{\pi} f(x)^2 \, dx \right]^{1/2} \left[\int_{-\pi}^{\pi} g(x)^2 \, dx \right]^{1/2}.$$

Complex Euclidean Space: $\mathbb{C}^n$

Points $Z = (z_1, \ldots, z_n)$, $W = (w_1, \ldots, w_n)$, where the z_j and w_j and the scalars c are complex numbers. There are almost no changes *except* that since we defined the absolute value of a complex number z so that $|z|^2 = z\bar{z}$ then we want the norm of Z to satisfy

$$|Z|^2 = z_1\overline{z_1} + z_2\overline{z_2} + \cdots + z_n\overline{z_n}.$$

Because we want $|Z|^2 = \langle Z,\, Z\rangle$, this leads us to define the inner product as

$$\langle Z,\, W\rangle = z_1\overline{w_1} + z_2\overline{w_2} + \cdots + z_n\overline{w_n}.$$

Thus the symmetry property becomes the slightly more complicated

$$\langle W,\, Z\rangle = \overline{\langle Z,\, W\rangle}$$

and thus $\langle Z, cW\rangle = \bar{c}\langle Z, W\rangle$. In addition

$$\begin{aligned}|Z+W|^2 =& |Z|^2 + \langle Z, W\rangle + \langle W, Z\rangle + |W|^2 \\ =& |Z|^2 + 2\text{Re}\{\langle Z, W\rangle\} + |W|^2.\end{aligned}$$

The Cauchy-Schwarz inequality remains true but the proof must be modified slightly because now the scalar t in our proof might need to be complex. In this case just use $0 \le |Z - tW|^2$ where $t = \overline{\langle Z, W\rangle} \,/\, |W|^2$.

[Last revised: September 6, 2014]