The Archimedean Property

Definition  An ordered field \( F \) has the Archimedean Property if, given any positive \( x \) and \( y \) in \( F \) there is an integer \( n > 0 \) so that \( nx > y \).

Theorem  The set of real numbers (an ordered field with the Least Upper Bound property) has the Archimedean Property.

This is the proof I presented in class. It is one of the standard proofs. The key is the following Lemma.

Lemma  The set \( \mathbb{N} \) of positive integers \( \mathbb{N} = \{0, 1, 2, \ldots\} \) is not bounded from above.

Proof  Reasoning by contradiction, assume \( \mathbb{N} \) is bounded from above. Since \( \mathbb{N} \subset \mathbb{R} \) and \( \mathbb{R} \) has the least upper bound property, then \( \mathbb{N} \) has a least upper bound \( \alpha \in \mathbb{R} \). Thus \( n \leq \alpha \) for all \( n \in \mathbb{N} \) and is the smallest such real number. Consequently \( \alpha - 1 \) is not an upper bound for \( \mathbb{N} \) (if it were, since \( \alpha - 1 < \alpha \), then \( \alpha \) would not be the least upper bound). Therefore there is some integer \( k \) with \( \alpha - 1 < k \). But then \( \alpha < k + 1 \). This contradicts that \( \alpha \) is an upper bound for \( \mathbb{N} \).

Proof of the Theorem  Since \( x > 0 \), the statement that there is an integer \( n \) so that \( nx > y \) is equivalent to finding an \( n \) with \( n > y/x \) for some \( n \). But if there is no such \( n \) then \( n < y/x \) for all integers \( n \). That is, \( y/x \) would be an upper bound for the integers. This contradicts the Lemma.

Remark  Homework Set 2 will have an example of an ordered field that does not have the Archimedean property.

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