Basic Definitions

In any metric space $S$:

- $S$ is **bounded** if it is contained in some ball in $\mathbb{R}^n$.
- $S$ is a **neighborhood** of $p$ if $S$ contains some open ball around $P$.
- A point $p$ is a **limit point** of $S$ if every neighborhood of $p$ contains a point $q \in S$, where $q \neq p$.
- If $p \in S$ is not a limit point of $S$, then it is called an **isolated point** of $S$.
- $S$ is **closed** if every limit point of $S$ is a point of $S$.
- A point $p \in S$ is an **interior point** of $S$ if $S$ contains a neighborhood of $p$.
- $S$ is **open** if every point of $S$ is an interior point of $S$.
- Let $S'$ denote all of the limit points of $S$. Then the **closure** $\bar{S}$ of $S$ is the set $S \cup S'$. It is the smallest closed set containing $S$ and is thus the intersection of all the closed sets containing $S$.
- A subset $T \subset S$ is **dense in $S$** if every point of $S$ is either in $T$ or a limit point of $T$ (or both).
• If $S$ is a metric space and $E \subset S$, let $E'$ be the limit points of $E$. Then the closure of $E = E \cup E'$. It is the smallest closed set that contains. It is also the intersection of all the closed sets that contain $E$.

• An open cover of $S$ is a family of open sets $T_\alpha \subset T$ with the property that every point of $S$ is in at least one of these open sets.

• A set $S \in \mathbb{R}$ with points $p$ has measure zero if given any $\epsilon > 0$ there is an open cover by open intervals $V_p$ so that

$$\sum_p \text{length of the } V_p < \epsilon.$$ 

The basic example is any countable set $S = \{x_1, x_2, \ldots\} \in \mathbb{R}$. Let $V_1$ be an open interval of length less than $\epsilon/2$ containing $x_1$, $V_2$ an open interval of length less than $\epsilon/2^2$ containing $x_2$, $\ldots V_k$ an open interval of length less than $\epsilon/2^k$ containing $x_k$, $\ldots$. Of course these intervals may overlap. However, since we have a geometric series,

$$\sum_k \text{length of the } V_k < \sum_{k=1}^{\infty} \frac{\epsilon}{2^k} = \epsilon.$$
A set $S$ is **compact** if every open cover of $S$ has a sub-cover consisting of a **finite** number of these open sets.

$E$ has the **Bolzano-Weierstrass property** if every infinite subset $x_1, x_2, \ldots$ of points in $E$ has at least one limit point $p$ in $E$.

In a metric space $X$ (or any “topological space”) a **separation** of $X$ is a pair $U$, $V$ of nonempty disjoint open subsets of $X$ whose union is $X$. The space $X$ is **connected** if a separation does not exist.

**Example:** The subset $(0, 2) \cup (2, 3)$ in $\mathbb{R}$ is not connected. The subset $(0, 2) \cup [2, 3)$ is connected.

**Remark:** For a subspace $Y$ of a larger topological space $X$ here is an alternate equivalent formulation (which the Rudin text uses).

If $Y$ is a subset of $X$, a **separation** of $Y$ is a pair of nonempty sets $A$ and $B$ whose union is $Y$, neither of which contains a limit point of the other. $Y$ is **connected** if no separation of $Y$ exists.

**Example** The following sets in the plane $\mathbb{R}^2$. 

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The $x-$axis and the graph of $y = 1/x$ for $x > 0$. It is not connected because neither piece contains a limit point of the other.