Basic Definitions

In any metric space S:

- S is *bounded* if it is contained in some ball in $\mathbb{R}^n$.
- S is a *neighborhood* of p if S contains some open ball around P.
- A point p is a *limit point* of S if every neighborhood of p contains a point $q \in S$, where $q \neq p$.
- If $p \in S$ is not a limit point of S, then it is called an *isolated point* of S.
- S is *closed* if every limit point of S is a point of S.
- A point $p \in S$ is an *interior point of* S if S contains a neighborhood of p.
- S is *open* if every point of S is an interior point of S.
- Let S' denote all of the limit points of S. Then the *closure* $\bar{S}$ *of* S is the set $S \cup S'$. It is the smallest closed set containing S and is thus the intersection of all the closed sets containing S.
- A subset $T \subset S$ is *dense in* S if every point of S is either in T or a limit point of T (or both).

- If S is a metric space and $E \subset S$, let E' be the limit points of E. Then the *closure* of $E = E \cup E'$. It is the smallest closed set that contains. It is also the intersection of all the closed sets that contain E.

- An *open cover* of S is a family of open sets $T_\alpha \subset T$ with the property that every point of S is in at least one of these open sets.

- A set $S \in \mathbb{R}$ with points p has *measure zero* if given any $\epsilon > 0$ there is an open cover by open intervals V_p so that

$$\sum_p \text{length of the } V_p < \epsilon.$$

The basic example is any countable set $S = \{x_1, x_2, \ldots\} \in \mathbb{R}$. Let V_1 be an open interval of length less than $\epsilon/2$ containing x_1, V_2 an open interval of length less than $\epsilon/2^2$ containing x_2, $\ldots V_k$ an open interval of length less than $\epsilon/2^k$ containing x_k, ... Of course these intervals may overlap. However, since we have a geometric series,

$$\sum_k \text{length of the } V_k < \sum_{k=1}^{\infty} \frac{\epsilon}{2^k} = \epsilon.$$

- A set S is *compact* if every open cover of S has a sub-cover consisting of a *finite* number of these open sets.
- E has the *Bolzano-Weierstrass property* if every infinite subset $x_1, x_2, \dots$ of points in E has at least one limit point p in E.
- In a metric space X (or any "topological space") a *separation* of X is s pair U, V ofnonempty disjoint open subsets of X whose union is X. The space X is *connected* if a separation does not exist.

  EXAMPLE: The subset $(0, 2) \cup (2, 3)$ in $\mathbb{R}$ is not connected. The subset $(0, 2) \cup [2, 3)$ is connected.

  REMARK: For a subspace Y of a larger topological space X here is an alternate equivalent formulation (which the Rudin text uses).

  If Y is a subset of X, a *separation* of Y is a pair of nonempty sets A and B whose union is Y, neither of which contains a limit point of the other. Y is *connected* if no separation of Y exists.

  EXAMPLE The following sets in the plane $\mathbb{R}^2$.

The $x-$axis and the graph of $y = 1/x$ for $x > 0$. It is not connected because neither piece contains a limit point of the other.