Basic Examples

$A = \{x \in \mathbb{R} : x = 1, 2, 3, 4\}$

$B_1 = \{x \in \mathbb{R} : 0 < x < 1\}$

$B_2 = \{(x,\ y) \in \mathbb{R}^2 : 0 < x < 1,\ y = 0\}$

$C = \{x \in \mathbb{R} : 0 \leq x \leq 1\}$

$D = \{x \in \mathbb{R} : x = 1, 2, 3, 4, \ldots\}$

$E = \{x \in \mathbb{R} : x = 1, 1/2, 1/3, \ldots\}$

$F = \{x \in \mathbb{R} : x = 1, 1/2, 1/3, \ldots\} \cup \{0\}$

$$G = \{x_n = (-1)^n + \frac{1}{n} \in \mathbb{R}, n = 1, 2, 3, \ldots\}$$

$H = \{(x, y) \in \mathbb{R}^2 : x \text{ and } y \text{ are positive integers}\}$

$I = \{(x, y) \in \mathbb{R}^2 : x + y > 1\}$

$J = \{(x, y) \in \mathbb{R}^2 : 0 < x^2 + y^2 < 4\}$

$$\begin{aligned} K = \{&e_1 = (1,\ 0,\ 0,\ 0),\ e_2 = (0,\ 1,\ 0,\ 0), \\ &e_3 = (0,\ 0,\ 1,\ 0,\ ),\ e_4 = (0,\ 0,\ 0,\ 1)\ \in \mathbb{R}^4\} \end{aligned}$$

$\ell_2 = \{$Sequences $x = (x_1,\, x_2,\, x_3,\, \ldots)$ where $x_j \in \mathbb{R}$ and $\displaystyle\sum_j |x_j|^2 < \infty\}$

Inner Product: $\langle x,\, y\rangle = x_1y_1 + x_2y_2 + x_3y_3 + \cdots$

Norm: $|x| = \sqrt{\sum_{i=1}^{\infty} |x_i|^2}$

This is Hilbert's *Hilbert Space*

L In ℓ_2 let $e_1 = (1,\, 0,\, 0,\, ,0, \ldots)$, $e_2 = (0,\, 1,\, 0,\, 0,\, \ldots)$, $e_3 = (0,\, 0,\, 1,\, 0,\, \ldots)$, $e_4 = (0,\, 0,\, 0,\, 1,\, \ldots)$, $\ldots$

Note if $i \neq j$ then $|e_i - e_j| = \sqrt{2}$. This set L is closed and bounded but not compact since if $0 < r < \sqrt{2}$ then the open balls $B_n = \{x \in \ell_2 : |x - e_n| < r\}$ cover this set but there is no finite sub-cover.

$\mathbb{Q} = \{x \in \mathbb{R} : x$ is a rational number$\}$

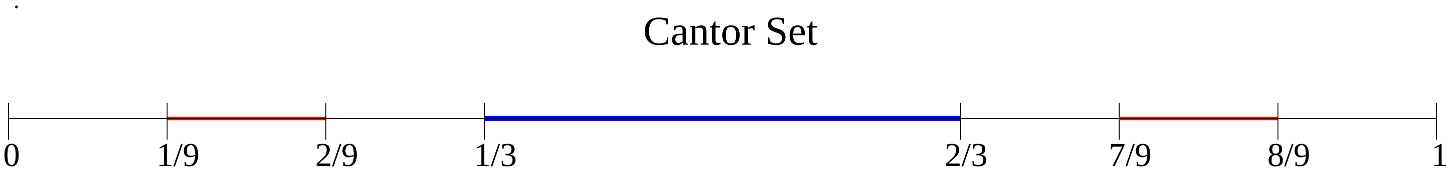
$\mathbb{Q}^2 = \{(x, y) \in \mathbb{R}^2 : x$ and y are rational numbers$\}$

The Cantor Set This surprising example was found in 1874 and has greatly influenced mathematics, particularly after the work of Cantor.

Begin with the interval $J = [0,\, 1]$. Divide it into 3 equal segments and delete the middle piece, $U_1 = (\frac{1}{3}, \frac{2}{3})$. Divide the remaining two intervals $[0,\, \frac{1}{3}]$

and $[\frac{2}{3}, 1]$ into three pieces and delete their middle pieces,

$$U_2 = (\tfrac{1}{9}, \tfrac{2}{9}) \cup (\tfrac{7}{9}, \tfrac{8}{9})$$



Divide the remaining four intervals into three pieces and delete their middle pieces:

$$U_3 = (\tfrac{1}{27}, \tfrac{2}{27}) \cup (\tfrac{7}{27}, \tfrac{8}{27}) \cup (\tfrac{19}{27}, \tfrac{20}{27}) \cup (\tfrac{25}{27}, \tfrac{26}{27})$$

Continue repeating this. The deleted set is

$$U := U_1 \cup U_2 \cup U_3 \cup U_4 \cup \cdots$$

The *Cantor Set* $K := J - U$ is what is left. Since each of the sets U_j are open, so is their union, U. There for K is a closed set. If you write the real numbers in $[0, 1]$ using base 3, then one only uses 0, 1, and 2 (much as base 10 we only use 0, 1, $\ldots$,9. the middle third intervals (Those in U) are precisely those whose base 3 representation have only 0's and 2's (no 1's). This set is is one-to-one correspondence with all the real numbers in $[0,1]$ written base 2. Thus K *is uncountable*.

We can compute the length of the U_j.

The length of U_1 is $1/3$
The length of U_2 is $2/9$
The length of U_3 is $4/27$
The length of U_k is $2^{k-1}/3^k$.

Therefore the length of U is the sum of the geometric series $\sum_{k=1}^{\infty} \frac{2^{k-1}}{3^k} = 1$. Since the length of J is 1, we conclude that the length of the Cantor set K is $1-1=0$. Thus K is an uncountable set with measure 0!.

Hardly intuitive.

[DISCRETE METRIC On *any* set S you can define a quite crude metruc, the *discrete metric*. For p and q in S, if $p \neq q$ define $d(p,q) = 1$ while if $p = q$ define $d(p,p) = 0$. The axioms for a matric are easy to verify.

Because the ball of radius $1/2$ centered at p only has the one point $\{p\}$, each set consisting of one point is open. Since every set is the union of one point sets, *every set is open*.

Because every point is isolated, there are no limit points. This implies that every set is closed (this also follows since every set is the complement of some set – and all sets are open – so their complements are closed.

The only compact sets are finite sets since most covers by open sets do not have a finite sub-cover.