Basic Examples

For each of these you should be able to determine if the set is:

finite  countable  bounded  open  closed  connected  compact

\[ A = \{ x \in \mathbb{R} : x = 1, 2, 3, 4 \} \]
\[ B_1 = \{ x \in \mathbb{R} : 0 < x < 1 \} \]
\[ B_2 = \{ (x, y) \in \mathbb{R}^2 : 0 < x < 1, \ y = 0 \} \]
\[ C = \{ x \in \mathbb{R} : 0 \leq x \leq 1 \} \]
\[ D = \{ x \in \mathbb{R} : x = 1, 2, 3, 4, \ldots \} \]
\[ E = \{ x \in \mathbb{R} : x = 1, 1/2, 1/3, \ldots \} \]
\[ F = \{ x \in \mathbb{R} : x = 1, 1/2, 1/3, \ldots \} \cup \{ 0 \} \]
\[ G = \{ x_n = (-1)^n + \frac{1}{n} \in \mathbb{R}, n = 1, 2, 3, \ldots \} \]
\[ H = \{ (x, y) \in \mathbb{R}^2 : x \text{ and } y \text{ are positive integers} \} \]
\[ I = \{ (x, y) \in \mathbb{R}^2 : x + y > 1 \} \]
\[ J = \{ (x, y) \in \mathbb{R}^2 : 0 < x^2 + y^2 < 4 \} \]
\[ K = \{ e_1 = (1, 0, 0, 0), \ e_2 = (0, 1, 0, 0), \]
\[ \quad e_3 = (0, 0, 1, 0), \ e_4 = (0, 0, 0, 1) \in \mathbb{R}^4 \} \]
\[ \ell_2 = \{ \text{sequences } x = (x_1, x_2, x_3, \ldots) \text{ where } x_j \in \mathbb{R} \text{ and } \sum_j |x_j|^2 < \infty \} \]

Inner product: \( \langle x, y \rangle = x_1y_1 + x_2y_2 + x_3y_3 + \cdots \),  \quad \text{Norm: } |x| = \sqrt{\sum_{i=1}^{\infty} |x_i|^2}.

This is Hilbert’s \textit{Hilbert Space}

L  In \ \ell_2 \ let \ e_1 = (1, 0, 0, 0, \ldots), \ e_2 = (0, 1, 0, 0, \ldots), \ e_3 = (0, 0, 1, 0, 0, \ldots), \ e_4 = (0, 0, 0, 1, \ldots), \ldots

If \( i \neq j \) then \( |e_i - e_j| = \sqrt{2}. \) This set \( L \) is closed and bounded but not compact since if \( 0 < r < \sqrt{2} \) then the open balls \( B_n = \{ x \in \ell_2 : |x - e_n| < r \} \) cover this set but there is no finite sub-cover.

\[ Q = \{ x \in \mathbb{R} : x \text{ is a rational number} \} \]
\[ Q^2 = \{ (x, y) \in \mathbb{R}^2 : x \text{ and } y \text{ are rational numbers} \} \]

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The Cantor Set This surprising example was found in 1874 and has greatly influenced mathematics, particularly after the work of Cantor.

Begin with the interval $J = [0, 1]$. Divide it into 3 equal segments and delete the middle piece, $U_1 = (\frac{1}{3}, \frac{2}{3})$. Divide the remaining two intervals $[0, \frac{1}{3}]$ and $[\frac{2}{3}, 1]$ into three pieces and delete their middle pieces,

$$U_2 = (\frac{1}{9}, \frac{2}{9}) \cup (\frac{7}{9}, \frac{8}{9})$$

Cantor Set

Divide the remaining four intervals into three pieces and delete their middle pieces:

$$U_3 = (\frac{1}{27}, \frac{2}{27}) \cup (\frac{7}{27}, \frac{8}{27}) \cup (\frac{19}{27}, \frac{20}{27}) \cup (\frac{25}{27}, \frac{26}{27})$$

Continue repeating this. The deleted set is

$$U := U_1 \cup U_2 \cup U_3 \cup U_4 \cup \cdots$$

The Cantor Set $K := J - U$ is what is left. Since each of the sets $U_j$ are open, so is their union, $U$. Therefore $K$ is a closed set. If you write the real numbers in $[0, 1]$ using base 3, then one only uses 0, 1, and 2 (much as base 10 we only use 0, 1, \ldots, 9. the middle third intervals (Those in $U$) are precisely those whose base 3 representation have only 0's and 2's (no 1's). This set is is one-to-one correspondence with all the real numbers in $[0, 1]$ written base 2. Thus $K$ is uncountable.

We can compute the length of the $U_j$.

- The length of $U_1$ is $1/3$
- The length of $U_2$ is $2/9$
- The length of $U_3$ is $4/27$
- The length of $U_k$ is $2^{k-1}/3^k$.

Therefore the length of $U$ is the sum of the geometric series $\sum_{k=1}^{\infty} \frac{2^{k-1}}{3^k} = 1$. Since the length of $J$ is 1, we conclude that the length of the Cantor set $K$ is $1 - 1 = 0$. Thus $K$ is an uncountable set with measure 0!.

Hardly intuitive.

[Discrete Metric] On any set $S$ you can define a quite crude metric, the discrete metric. For $p$ and $q$ in $S$, if $p \neq q$ define $d(p, q) = 1$ while if $p = q$ define $d(p, p) = 0$. The axioms for a metric are easy to verify.
Because the ball of radius 1/2 centered at $p$ only has the one point $\{p\}$, each set consisting of one point is open. Since every set is the union of one point sets, *every set is open*. Every set is bounded, in fact, it is in a closed ball ball of radius 1 centered at any point $p$.

Because every point is isolated, there are no limit points. This implies that every set is closed (this also follows since every set is the complement of some set – and all sets are open – so their complements are closed.

The only compact sets are finite sets since most covers by open sets do not have a finite sub-cover. This thus gives examples of metric spaces with closed and bounded sets that are not compact.