Compactness

In these notes we will assume all sets are in a metric space X. These proofs are merely a rephrasing of this in Rudin – but perhaps the differences in wording will help.

Intuitive remark: a set is compact if it can be guarded by a finite number of arbitrarily nearsighted policemen.

Theorem *A compact set K is bounded.*

PROOF Pick any point $p \in K$ and let $B_n(p) = \{x \in K : d(x,p) < n\}$, $n = 1, 2, \ldots$. These open balls cover K. By compactness, a finite number also cover K. The largest of these is a ball that contains K.

Theorem 2.34 *A compact set K is closed.*

PROOF We show that the complement $K^c = X - K$ is open. Pick a point $p \notin K$. If $q \in K$, let V_q and W_q be open balls around p and q of radius $\frac{1}{2}d(p,q)$. Observe that if $x \in W_q$ then

$$d(q,p) \le d(q,x) + d(x,p) < \tfrac{1}{2}d(p,q) + d(x,p)$$

so $d(x,p) > \frac{1}{2}d(p,q)$, that is, all the points in this

ball are at least $\frac{1}{2}d(p,q)$ from p .
By compactness, a finite number of them, $W_{q_1}, \ldots W_{q_N}$ cover K. Look at the corresponding balls $V_{q_1}, \ldots V_{q_N}$. They are all centered at p. The smallest (their intersection) is a neighborhood of p that contains no points of K.

Theorem 2.35 Closed subsets of compact sets are compact.

Proof Say $F \subset K \subset X$ where F is closed and K is compact. Let $\{V_\alpha\}$ be an open cover of F. Then F^c is a trivial open cover of F^c. Consequently $\{F^c\} \cup \{V_\alpha\}$ is an open cover of K. By compactness of K it has a finite sub-cover – which gives us a finite sub-cover of F.

Theorem 2.38 Let I_n be a sequence of nested closed intervals in $\mathbb{R}$, so $I_n \supseteq I_{n+1}$, $n = 1, 2, \ldots$. Then $\cap_{n=1}^{\infty} I_n$ is not empty.
Proof Say $I_n = \{x \in \mathbb{R} : a_n \leq x \leq b_n\}$. The nested property means

$$a_1 \leq a_n \leq a_{n+1} \leq b_{n+1} \leq b_n \leq b_1.$$

Let $a = \sup a_n$ and $b = \inf b_n$. It is clear that $\cap_{n=1}^{\infty} I_n = \{a \leq x \leq b\}$.

It is clear that this immediately extends to closed cells ("rectangles") in $\mathbb{R}^2$ and $\mathbb{R}^k$. We use it to show

Theorem 2.40 Closed and bounded intervals $x \in \mathbb{R} : \{a \leq x \leq b\}$ are compact.

PROOF Idea: keep on dividing $a \leq x \leq b$ in half and use a microscope.

Say there is an open cover $\{G_\alpha\}$ that has no finite sub-cover. Divide the interval in half. Then one (or both) halves are closed sets with an open cover that has no finite cover. Keep on repeating this. At the n^{th} step we have a closed interval I_n of length $(b - a)/2^n$ where there is no finite sub-cover of our $\{G_\alpha\}$. By the previous theorem, the intersection of these (nested) intervals $\cap_{n=1}^{\infty} I_n$ has at point p. Since p is contained in at least one of the $\{G_\alpha\}$ so there is some interval around p. This shows that for n large I_n is covered by one of the sets G_α. Contradiction.

Theorem 2.37 In any metric space, an infinite subset E of a compact set K has a limit point in K. [Bolzano-Weierstrass]

PROOF Say no point of K is a limit point of E. Then each point of K would have a neighborhood containing at most one point q of E. A finite number of these neighborhoods cover K – so the set E must be finite.

Theorem 2.41 Let $\{E \in \mathbb{R}^k\}$. The following properties are equivalent:
(a) E is closed and bounded.
(b) E is compact.
(c)Every infinite subset of E has a limit point in E. [Bolzano-Weierstrass Property]
PROOF We do this for sets $E \in \mathbb{R}^1$. The ore general case is then straightforward.
(a) implies (b): Since E is bounded it is contained in some closed interval I. This interval is compact (Theorem 2.40). But then E is a closed subset of a compact set so it is compact (Theorem 2.35).
(b) implies (c): Theorem 2.37.
(c) implies (a). If E is not bounded, then for each $n = 1, 2, \ldots$ there is a point $x_n \in E$ with $|x_n| > n$. This infinite set has no limit point, a contradiction.
If $E \subset \mathbb{R}$ is not closed then there is a point $p \in \mathbb{R}$ which is a limit point of E but not in E. Thus, for each $n = 1, 2, 3, \ldots$ there is a point $x_n \in E$ with $|x_n - p| < 1/n$. This set $S = \{x_1, x_2, \ldots\}$ has

$p \notin E$ as its only limit point. Contradiction.

Example Let K be a compact set in a metric space X and let $p \in X$ but $p \notin K$. Then there is a point x_0 in K that is closest to p. In other words, let $\alpha = \inf_{x \in K} d(x, p)$. then there is at least one point $x_0 \in K$ with $d(x_0, p) = \alpha$,

Remark: There may be many such points, for example if K is the unit circle $x^2 + y^2 = 1$ in the plane and $p = (0, 0)$, then every point on the circle minimizes the distance to the origin.

Solution: For any $n = 1, 2, \ldots$ there is at least one point $x_n \in K$ with $d(x_n, p) \leq \alpha + \frac{1}{n}$. If this set $\{x_1, x_2, \ldots\}$ is finite (for instance if K only has a finite number of points), pick the point closest to p. If the set has infinite many points, by the Bolzano-Weierstrass property it has a limit point q in K. This is the desired point in K that is closest to p.

Example In ℓ_2 the set of unit vectors $e_1 = (1, 0, 0, \ldots)$, $e_2 = (0, 1, 0, 0 \ldots)$, $\ldots$ is closed and bounded but not compact.