Differentiate Limit $f_n \to f$

Say f_n are a sequence of smooth functions that converge to some function $f(x)$. Is f differentiable and does f_n' converge to $f'(x)$?

Sometimes, sometimes not.

Here is one (standard) theorem that is useful.

Theorem *Let $f_n \in C^1([a,b])$. Assume*

(i). For some $x_0 \in [a,b]$ the sequence $f_n(x_0)$ converges, say $f_n(x_0) \to c$.

(ii). The $f_n'(x)$ converge uniformly to some function, say $g(x) \in C([a,b])$ in $[a,\, b]$.

Then there is a function $f \in C^1([a,b])$ so that $f_n \to f$ uniformly in $[a,b]$ with $f(x_0) = c$ and $f'(x) = g(x)$ for $x \in [a,b]$.

PROOF For $x \in [a,b]$ the obvious candidate for f is

$$f(x) = c + \int_{x_0}^{x} g(t)\, dt.$$

It clearly satisfies $f(x_0) = c$, is in $C^1([a,b])$, and, by the Fundamental Theorem of Calculus satisfies $f'(x) = g(x)$. We need only show that f_n converges uniformly to this f.

By the Fundamental Theorem of Calculus, for $x \in [a,b]$

$$f_n(x) = f_n(x_0) + \int_{x_0}^{x} f_n'(t)\, dt.$$

Therefore

$$f_n(x) - f(x) = [f_n(x_0) - c] + \int_{x_0}^{x} [f_n'(t) - g(t)]\, dt.$$

But given $\epsilon > 0$ there are integers N_1 and N_2 so that if $n \ge N_1$ then $|f_n(x_0) - c| < \epsilon$ while if $n \ge N_2$ then using the supremum norm on $[a,b]$ then $\|f_n' - g\| < \epsilon$. Thus, if $n \ge \max\{N_1, N_2\}$ then

$$|f_n(x) - f(x)| \le \epsilon + \epsilon(b - a).$$

Because the right hand side holds for all $x \in [a,b]$, this proves the uniform convergence of f_n to f in the interval $[a,b]$.