Logic Notation: Convergence and Continuity

Reference: This is copied from the book *Fundamentals of Abstract Analysis* by Andrew Gleason.

Convergence of a Sequence

1. A sequence x_n of real numbers is said to be *increasing* (or *monotone increasing*) if

$$(\forall m, n)\ (m > n) \Rightarrow x_m \geq x_n.$$

2. Let z_n be a sequence of complex numbers. The sequence is said to *converge to* z, in symbols, $z_n \to z$, if

$$(\forall \epsilon > 0)\ (\exists N \in \mathbb{N})\ (\forall n > N)\ |z_n - z| < \epsilon.$$

3. Let z_n be a sequence of complex numbers. The sequence is said to *converge* if

$$(\exists z \in \mathbb{C})\ (\forall \epsilon > 0)\ (\exists N \in \mathbb{N})\ (\forall n > N)\ \ |z_n - z| < \epsilon.$$

In greater detail:
Since this begins with an existential quantifier, we must move first by choosing a number z. Since the next quantifier is universal, our opponent moves next by choosing a positive number ϵ. The opponent will presumably make the best possible move and choose ϵ so that

$$(\exists N \in \mathbb{N})\ (\forall n > N)\ \ |z_n - z| < \epsilon$$

is false (if possible). Now it is our move to choose a number N with a knowledge of the previous moves, that is, N may (and surely will) depend on both z and ϵ. Finally our opponent chooses a number $n > N$ and the burden is on us to prove the inequality $|z_n - z| < \epsilon$.

Chess: Checkmate

4. Using this language, for a chess game, the usual “white mates in two moves” can be thought of as:

(∃ white move) (∀ black moves) (∃ white move) (∀ black moves) black is checkmated

where of course “black is checkmated” means “white can capture black’s king”.

Continuous Functions

5. Let S and T be metric spaces with metrics $d_S(\cdot, \cdot)$ and $d_T(\cdot, \cdot)$ and $f : S \to T$. Then f is *continuous at a point* $p \in S$ if

$$(\forall \epsilon > 0)\ (\exists \delta > 0)\ (\forall q \in S)\ d_S(p, q) < \delta \Rightarrow d_T(f(p), f(q)) < \epsilon.$$

f is continuous on the whole set S if it is continuous at each point of S. More formally,

$$(\forall p \in S)\ (\forall \epsilon > 0)\ (\exists \delta > 0)(\forall q \in S)\ \ d_S(p, q) < \delta \Rightarrow d_T(f(p), f(q)) < \epsilon.$$

We restate this in terms of sequences. Say $p \in S$. Then f is continuous at p if and only for every sequence $x_n \to p$ then $f(x_n) \to f(p)$. That is

$$\lim f(x_n) = f(\lim x_n)$$

We can also restate the definition of continuity in terms of balls $B_S(p, \delta)$ and $B_T(s, \epsilon)$ in S and T, respectively:

$$(\forall p \in S)\ (\forall \epsilon > 0)\ (\exists \delta > 0)\ \ f(B_S(p, \delta)) \subseteq B_T(f(p), \epsilon).$$

5'. Equivalent Definition of Continuity

It is interesting that one can describe "continuity" in a dramatically different way without using "limit" or explicitly referring to the metric. As a consequence, this is used as the *definition* of continuity in more general topological spaces that are not metric spaces. It also simplifies many proofs.

Some notation: Let $f : S \to T$. If S_0 is a subset of S, denote by $f(S_0)$ the set of all image points of S_0 under the function f, so

$$f(S_0) = \{t \in T : t = f(s) \text{ for some } s \in S_0\}.$$

Similarly, if T_0 is a subset of T, denote by $f^{-1}(T_0)$ the set of all points in S whose image is in T_0:

$$f^{-1}(T_0) = \{s \in S : f(s) \in T_0\}.$$

$f^{-1}(T_0)$ is called the *preimage* of T_0. It can happen that no points in S have their image in T_0. A simple example is the map $f : \mathbb{R} \to \mathbb{R}$ defined by $f(s) = s^2$ and $T_0 = \{t \in \mathbb{R} : t < 0\}$. Then the preimage of this T_0 is empty since the square of any real number is not negative.

Caution The operation f^{-1} applied to subsets of T behaves nicely. It preserves inclusions, unions, intersections and differences of sets. However the operation f applied to subsets of S is more complicated.

Note also that if $f : S \to T$, while

$$f^{-1}(f(S_0)) \supset S_0 \ \text{ for } \ S_0 \subset S \qquad \text{and} \qquad f(f^{-1}(T_0)) \subset T_0 \ \text{ for } \ T_0 \subset T,$$

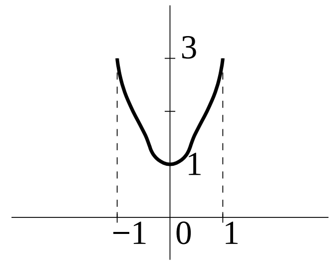
equality often does not hold. Here is a (non-pathological) example:

Let $f : \mathbb{R} \to \mathbb{R}$ be $f(x) = 2x^2 + 1$ and use the standard notation $[a, b] = \{a \le x \le b\}$. Since f is not one-to-one, then two different sets can have the same images

$$f([0,\, 1]) = f([-1,\, 1]) = [1,\, 3].$$

while because f is not onto, two different sets can have the same preimages

$$f^{-1}([0,\, 3]) = f^{-1}([1,\, 3]) = [-1,\, 1],$$



$f(x) = 2x^2 + 1$

Using these we obtain the examples:

$$f^{-1}(f([0,\, 1])) = f^{-1}([1,\, 3]) = [-1,\, 1] \qquad \text{and} \qquad f(f^{-1}([0, 3])) = f([-1,\, 1]) = [1,\, 3].$$

Theorem *Let S, T be metric spaces and $f : S \to T$. Then f is continuous on $S \iff$ for any open set $G \subset T$, the set $f^{-1}(G)$ is open. That is, the preimage of an open set is open.*

Proof: $\Longrightarrow$ Say f is continuous and we are given an open set $G \in T$. If $f^{-1}(G)$ is the empty set, there is nothing to prove. Thus, say $p \in f^{-1}(G)$. This means there is a point $q \in G$ with $f(p) = q$. We need to find a ball $U := B_S(p,\, \delta)$ around p so that $f(U) \subset G$. Since G is open, it contains some ball $V := B_T(q,\, \epsilon)$. By the continuity of f there is a $\delta > 0$ so that $f(U) \subset V$. Thus the open set U is in the preimage of G. $\square$

$\Longleftarrow$. Say the preimage of any open set $G \subset T$ is open. To show that f is continuous at every point $p \in S$. Given any $\epsilon > 0$, let $G := B_T(f(p),\, \epsilon)$. We need to find a $\delta > 0$ so that image of $B_S(p,\, \delta) \subset B_T(f(p),\, \epsilon)$. But since the preimage of G is open, it contains some small ball $B_S(p,\, \delta)$ around p. $\square$

Using that a set is closed if and only if its complement is open and that $f^{-1}(T_0^c) = [f^{-1}(T_0)]^c$ for every subset $T_0 \subset T$, we can use closed sets instead of open sets to varify continuity. That is,

Corollary *Let S, T be metric spaces and $f : S \to T$. Then f is continuous on $S \iff$ for any closed set $C \subset T$, the set $f^{-1}(C)$ is closed. That is, the preimage of a closed set is closed.*

6. In general if f is continuous at every point p of a metric space, the choice of δ will depend on both ϵ and the particular point p. If given ϵ we can find a δ that works simultaneously for every point p then the function is said to be *uniformly continuous*. More formally

$$(\forall \epsilon > 0)\ (\exists \delta > 0)\ (\forall p,\, q \in S)\ \ d_S(p,\, q) < \delta \ \Rightarrow\ d_T(f(p),\, f(q)) < \epsilon).$$

Equivalently:

$$(\forall \epsilon > 0)\ (\exists \delta > 0)\ (\forall p \in S)\ \ f(B_S(p,\, \delta)) \subseteq B_T(f(p),\, \epsilon).$$

Convergence of a Sequence of Functions

If S is a set, (T, d) is a metric space and $\{f_n\} : S \to T$ is a sequence of functions, the next two definitions concern the convergence of the $\{f_n\}$ to a function g.

7. $\{f_n\}$ is said to converge *pointwise* to g if for all $p \in S$ we have $f_n(p) \to g(p)$. In greater detail

$$(\forall p \in S)\ (\forall \epsilon > 0)\ (\exists N \in \mathbb{N})(\forall n > N)\ \ d(f_n(p),\, g(p)) < \epsilon.$$

9.. $\{f_n\}$ is said to converge to g *uniformly on* S if

$$(\forall \epsilon > 0)\ (\exists N \in \mathbb{N})\ (\forall n > N)\ (\forall p \in S)\ \ d(f_n(p),\, g(p)) < \epsilon.$$

For pointwise convergence the choice of N can depend on p, while for uniform convergence the same N works simultaneously for all $p \in S$.
For example, if $S = \{0 < x < 1\}$ then $f_n(x) := x^n$ converges pointwise but not uniformly to $g(x) := 0$. However if $S := \{0 < x < 1/2\}$ then f_n does converge uniformly to 0.

[Last revised: November 11, 2014]