

Problem Set 1

DUE: Tues. Sept. 9, 2014. *Late papers will be accepted until 1:00 PM Wednesday.*

\LaTeX If you will be writing many documents that contain equations, it is wise to learn (and use) \LaTeX . It is available on Windows, Macs, and Linux – and is FREE. See <http://www.math.upenn.edu/~kazdan/202F13/tex/>. For some students, this might be the most useful item you learn in this course.

This week. Please read all of Chapter 1 and Sections 2.1-2.14 in Chapter 2. of the Rudin text.

1. (Rudin, p. 21 #1) If $r \neq 0$ is rational and x is irrational, prove that $r + x$ and rx are irrational.
2. (Rudin, p. 22 #2) Prove that there is no rational number whose square is 12.
3. (Rudin, p. 22 #3) Prove Proposition 1.15, i.e., show that the axioms of multiplication imply the following statements:
 - a) If $x \neq 0$ and $xy = xz$ then $y = z$.
 - b) If $x \neq 0$ and $xy = x$ then $y = 1$.
 - c) If $x \neq 0$ and $xy = 1$, then $y = 1/x$.
 - d) If $x \neq 0$ then $1/(1/x) = x$.

4. (Rudin, p. 23 #17) For $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ with the Euclidean distance, prove that

$$|\mathbf{x} + \mathbf{y}|^2 + |\mathbf{x} - \mathbf{y}|^2 = 2|\mathbf{x}|^2 + 2|\mathbf{y}|^2$$

and interpret this geometrically as a statement about a parallelogram.

HINT: Use

$$|\mathbf{x} + \mathbf{y}|^2 = (\mathbf{x} + \mathbf{y}) \cdot (\mathbf{x} + \mathbf{y}) = \mathbf{x} \cdot \mathbf{x} + 2\mathbf{x} \cdot \mathbf{y} + \mathbf{y} \cdot \mathbf{y}.$$

5. (Rudin, p. 23 #19) Suppose \mathbf{a} , \mathbf{b} , and \mathbf{x} are points in \mathbb{R}^k . Show that

$$|\mathbf{x} - \mathbf{a}| = 2|\mathbf{x} - \mathbf{b}|$$

is satisfied if and only if \mathbf{x} lies on some sphere $|\mathbf{x} - \mathbf{c}| = r$. [You should find $\mathbf{c} \in \mathbb{R}^k$ and $r > 0$ in terms of \mathbf{a} and \mathbf{b}]. [HINT: See the Hint for Problem 4]

Generalization: For real $\lambda > 0$, $\lambda \neq 1$, consider the points $\mathbf{x} \in \mathbb{R}^k$ that satisfy

$$|\mathbf{x} - \mathbf{a}| = \lambda|\mathbf{x} - \mathbf{b}|$$

Show that these points lie on a sphere. (Thus find the center and radius of this sphere in terms of \mathbf{a} , \mathbf{b} and λ . [You can reduce the case $\lambda > 1$ to the case $0 < \lambda < 1$ by letting $\lambda = 1/\mu$].

What if $\lambda = 1$?

6. Let S_1, S_2, S_3, \dots be finite sets and $A = \cup_{j=1}^{\infty} S_j$. Show that A is countable (or finite). In other words, a countable union of finite sets is countable (or finite). [A is finite, for instance, if all the sets S_j are subsets of, say, S_1].
7. (Rudin, p.43 #2) A complex number is *algebraic* if it is a root of a polynomial $a_0z^n + \dots + a_n$ whose coefficients are all integers. Prove that the set of all algebraic numbers is countable. [HINT: For every positive integer N there are only finitely many equations with $n + |a_0| + \dots + |a_n| = N$.]
8. (Rudin, p. 22 #6) The point of this problem is, for any real $b > 1$ and any real x to define b^x . So far we can only compute b^x for the special cases $x = n$ and for $x = 1/n$, where $n = 1, 2, \dots$. First we extend this to rational x and then to all real x .
Fix $b > 1$. Let m, n, p, q be integers with $n > 0$, $q > 0$. Set $r = m/n = p/q$.
 - a) Prove that $(b^m)^{1/n} = (b^p)^{1/q}$. Thus, it makes sense to define $b^r = (b^m)^{1/n}$.
 - b) If r and s are rational, prove that $b^{r+s} = b^r b^s$.
 - c) If x is real, define $B(x)$ to be the set of all numbers b^t , where t is rational and $t \leq x$. Prove that for r rational

$$b^r = \sup B(r).$$

Hence it makes sense to *define* $b^x = \sup B(x)$ for all real x .

- d) With this definition, prove that for all real x, y : $b^{x+y} = b^x b^y$.
9. (Rudin, P.43 #3) Prove there exist real numbers that are not algebraic.
10. (Rudin, P.43 #4) Is the set of all irrational real numbers countable? Why?

[Last revised: September 8, 2014]