Math 508, Fall 2014 **Jerry L. Kazdan**

Problem Set 10

DUE: Thurs. Nov. 20, 2014. *Late papers will be accepted until 1:00 PM Friday.*

This week. Please read the material
`http://www.math.upenn.edu/~kazdan/508F14/Notes/K-F-contracting-maps.pdf`
on contracting mappings.

1. Let S and T be linear spaces and $L : S \to T$ be a linear map. Say V_1 and V_2 are (distinct!) solutions of the equation $LX = Y_1$ while W is a solution of $LX = Y_2$. Answer the following in terms of V_1, V_2, and W.
   a) Find some solution of $LX = 2Y_1 - 7Y_2$.
   b) Find another solution (other than W) of $LX = Y_2$.

2. Let $f(x) \in C([a, b])$. Show that
$$\exp\left[\frac{1}{b-a}\int_a^b f(x)\,dx\right] \le \frac{1}{b-a}\int_a^b \exp[f(x)]\,dx$$
   [HINT: Use the inequality $e^u \ge 1 + u$ where $u = f - \bar{f}$. Here $\bar{f}$ = average of f = $\frac{1}{b-a}\int_a^b f(x)\,dx$.]

3. Let f be continuous on the interval $[0, 1]$. Show that $\displaystyle\lim_{n\to\infty}\int_0^1 f(x)\sin nx\,dx = 0.$

4. Determine which of the following function sequences of functions converge pointwise or uniformly:
   a) $f_n(x) = \frac{\sin x}{n}$, $x \in \mathbb{R}$
   b) $f_n(x) = \frac{1}{1+nx}$, $x \in [0,\ 1]$
   c) $f_n(x) = \frac{x}{1+nx^2}$, $x \in \mathbb{R}$.

5. a) If $f_n \to f$ and $g_n \to g$ pointwise in $C([0,\ 1])$. is it true that $f_n \cdot g_n \to f \cdot g$ pointwise?
   b) If $f_n \to f$ and $g_n \to g$ uniformly in $C([0,\ 1])$, is it true that $f_n \cdot g_n \to f \cdot g$ uniformly?

6. a) If $f : [0,\ 1] \to \mathbb{R}$ is a continuous function with the property that $\int_0^1 f(x)g(x)\,dx = 0$ for *all* continuous functions g, prove that $f(x) = 0$ for all $x \in [0,\ 1]$.
   b) If $f : [0,\ 1] \to \mathbb{R}$ is a continuous function with the property that $\int_0^1 f(x)g(x)\,dx = 0$ for *all* C^1 functions g that satisfy $g(0) = g(1) = 0$, must it be true that $f(x) = 0$ for all $x \in [0,\ 1]$? Proof or counterexample.

7. The maps (a) $x \mapsto x + \frac{1}{x}: \ [1, \infty] \to [1, \infty]$, (b) $x \mapsto \frac{x}{2}: \ (0, 1] \to (0, 1]$ have no fixed points.
Which conditions of the Contracting Map Theorem are not satisfied in these examples?

8. Consider $f(x) := \sum_{k=1}^{\infty} \frac{\sin kx}{1+k^4}$.
   a) For which real x is f continuous?
   b) Is f differentiable? Why?

9. Let a_n be a bounded sequence of complex numbers and $f(z) = \sum_{n=1}^{\infty} \frac{a_n}{n^z}$, where $z = x+iy$.
If $c > 1$, show that this series converges absolutely and uniformly in the half-plane $\{z = x + iy \in \mathbb{C} \,|\, x \geq c\}$.

10. Show that the sequence of functions $f_n(x) := n^3 x^n (1-x)$ does not converge uniformly on $[0, 1]$.

11. For each of the following give an example of a sequence of continuous functions. Justify your assertions. [A clear sketch may be adequate — as long as it is convincing].
   a) $f_n(x)$ that converge to zero at every x, $0 \leq x \leq 1$, but *not* uniformly.
   b) $g_n(x)$ that converge to zero at every x, $0 \leq x \leq 1$, but $\int_0^1 g_n(x)\,dx \geq 1$.
   c) $h_n(x)$ converge to zero uniformly for $0 \leq x < \infty$, but $\int_0^\infty h_n(x)\,dx \geq 1$.

Bonus Problem

[Please give this directly to Professor Kazdan]

B-1 [Hölder's inequality] Let p, $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$. In class we proved that

$$st \leq \frac{s^p}{p} + \frac{t^q}{q} \qquad \text{for all} \quad s, t > 0.$$

a) Use this to show that for any complex numbers a_k, b_k

$$\sum_{k=1}^{n} |a_k b_k| \leq \left[\sum_{k=1}^{n} |a_k|^p\right]^{1/p} \left[\sum_{k=1}^{n} |b_k|^q\right]^{1/q}.$$

[Suggestion: First do the special case $\left[\sum_{k=1}^{n} |a_k|^p\right]^{1/p} = 1$ and $\left[\sum_{k=1}^{n} |b_k|^q\right]^{1/q} = 1$. Then reduce the general case to this special case.] If $p = q = 1/2$ this is the Schwarz inequality.

b) Similarly, show that for any continuous functions f, g

$$\int_a^b |f(x)g(x)|\,dx \le \left[\int_a^b |f(x)|^p\,dx\right]^{1/p} \left[\int_a^b |g(x)|^q\,dx\right]^{1/q}.$$

c) Let p, $q > 1$ with $\dfrac{1}{p} + \dfrac{1}{q} = 1$. and let $X := (x_1, \ldots, x_n) \in \mathbb{R}^n$ and $f \in C([a,b])$. Use Hölder's inequality (above) to prove the triangle inequality for the norms

$$\|X\|_p := \left[\sum_{k=1}^{n} |x_k|^p\right]^{1/p} \quad \text{and} \quad \|f\|_p := \left[\int_a^b |f(x)|^p\,dx\right]^{1/p}.$$

B-2 (For those who have studied rings). Let $\mathcal{C}$ be the ring of continuous functions on the interval $0 \le x \le 1$.

a) If $0 \le c \le 1$, show that the subset $\{\, f \in \mathcal{C} \mid f(c) = 0 \,\}$ is a maximal ideal.

b) Show that *every* maximal ideal in $\mathcal{C}$ has this form.

[Last revised: November 14, 2014]