Problem Set 10
Due: Thurs. Nov. 20, 2014. Late papers will be accepted until 1:00 PM Friday.

This week. Please read the material on contracting mappings.

1. Let $S$ and $T$ be linear spaces and $L : S \to T$ be a linear map. Say $V_1$ and $V_2$ are (distinct!) solutions of the equation $LX = Y_1$ while $W$ is a solution of $LX = Y_2$. Answer the following in terms of $V_1$, $V_2$, and $W$.
   a) Find some solution of $LX = 2Y_1 - 7Y_2$.
   b) Find another solution (other than $W$) of $LX = Y_2$.

2. Let $f(x) \in C([a, b])$. Show that
   \[
   \exp \left[ \frac{1}{b-a} \int_a^b f(x) \, dx \right] \leq \frac{1}{b-a} \int_a^b \exp[f(x)] \, dx
   \]
   [Hint: Use the inequality $e^u \geq 1 + u$ where $u = f - \bar{f}$. Here $\bar{f}$ = average of $f = \frac{1}{b-a} \int_a^b f(x) \, dx$.]

3. Let $f$ be continuous on the interval $[0, 1]$. Show that $\lim_{n \to \infty} \int_0^1 f(x) \sin nx \, dx = 0$.

4. Determine which of the following function sequences of functions converge pointwise or uniformly:
   a) $f_n(x) = \sin \frac{x}{n}$, $x \in \mathbb{R}$
   b) $f_n(x) = \frac{1}{1+nx}$, $x \in [0, 1]$
   c) $f_n(x) = \frac{x}{1+nx^2}$, $x \in \mathbb{R}$.

5. a) If $f_n \to f$ and $g_n \to g$ pointwise in $C([0, 1])$, is it true that $f_n \cdot g_n \to f \cdot g$ pointwise?
   b) If $f_n \to f$ and $g_n \to g$ uniformly in $C([0, 1])$, is it true that $f_n \cdot g_n \to f \cdot g$ uniformly?

6. a) If $f : [0, 1] \to \mathbb{R}$ is a continuous function with the property that $\int_0^1 f(x)g(x) \, dx = 0$ for all continuous functions $g$, prove that $f(x) = 0$ for all $x \in [0, 1]$.
   b) If $f : [0, 1] \to \mathbb{R}$ is a continuous function with the property that $\int_0^1 f(x)g(x) \, dx = 0$ for all $C^1$ functions $g$ that satisfy $g(0) = g(1) = 0$, must it be true that $f(x) = 0$ for all $x \in [0, 1]$? Proof or counterexample.
7. The maps (a) \( x \mapsto x + \frac{1}{x} : [1, \infty] \to [1, \infty] \), (b) \( x \mapsto \frac{x}{2} : (0, 1] \to (0, 1] \) have no fixed points.
Which conditions of the Contracting Map Theorem are not satisfied in these examples?

8. Consider \( f(x) := \sum_{k=1}^{\infty} \frac{\sin kx}{1 + k^4} \).
   a) For which real \( x \) is \( f \) continuous?
   b) Is \( f \) differentiable? Why?

9. Let \( a_n \) be a bounded sequence of complex numbers and \( f(z) = \sum_{n=1}^{\infty} \frac{a_n}{n^z} \), where \( z = x + iy \).
   If \( c > 1 \), show that this series converges absolutely and uniformly in the half-plane \( \{z = x + iy \in \mathbb{C} \mid x \geq c\} \).

10. Show that the sequence of functions \( f_n(x) := n^3 x^n(1 - x) \) does not converge uniformly on \([0, 1]\).

11. For each of the following give an example of a sequence of continuous functions. Justify your assertions. [A clear sketch may be adequate — as long as it is convincing].
   a) \( f_n(x) \) that converge to zero at every \( x, 0 \leq x \leq 1 \), but not uniformly.
   b) \( g_n(x) \) that converge to zero at every \( x, 0 \leq x \leq 1 \), but \( \int_0^1 g_n(x) \, dx \geq 1 \).
   c) \( h_n(x) \) converge to zero uniformly for \( 0 \leq x < \infty \), but \( \int_0^\infty h_n(x) \, dx \geq 1 \).

**Bonus Problem**

[Please give this directly to Professor Kazdan]

B-1 [HÖLDER’S INEQUALITY] Let \( p, q > 1 \) with \( \frac{1}{p} + \frac{1}{q} = 1 \). In class we proved that
\[
st \leq \frac{s^p}{p} + \frac{t^q}{q}
\]
for all \( s, t > 0 \).

a) Use this to show that for any complex numbers \( a_k, b_k \)
\[
\sum_{k=1}^{n} |a_k b_k| \leq \left[ \sum_{k=1}^{n} |a_k|^p \right]^{1/p} \left[ \sum_{k=1}^{n} |b_k|^q \right]^{1/q}.
\]
[SUGGESTION: First do the special case \( \left[ \sum_{k=1}^{n} |a_k|^p \right]^{1/p} = 1 \) and \( \left[ \sum_{k=1}^{n} |b_k|^q \right]^{1/q} = 1 \). Then reduce the general case to this special case.] If \( p = q = 1/2 \) this is the Schwarz inequality.

2
b) Similarly, show that for any continuous functions $f, g$

$$\int_a^b |f(x)g(x)| \, dx \leq \left[ \int_a^b |f(x)|^p \, dx \right]^{1/p} \left[ \int_a^b |g(x)|^q \, dx \right]^{1/q}.$$

c) Let $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, and let $X := (x_1, \ldots, x_n) \in \mathbb{R}^n$ and $f \in C([a, b])$. Use Hölder’s inequality (above) to prove the triangle inequality for the norms

$$\|X\|_p := \left[ \sum_{k=1}^n |x_k|^p \right]^{1/p} \quad \text{and} \quad \|f\|_p := \left[ \int_a^b |f(x)|^p \, dx \right]^{1/p}.$$

B-2 (For those who have studied rings). Let $\mathcal{C}$ be the ring of continuous functions on the interval $0 \leq x \leq 1$.

a) If $0 \leq c \leq 1$, show that the subset \{ $f \in \mathcal{C} \mid f(c) = 0$ \} is a maximal ideal.

b) Show that every maximal ideal in $\mathcal{C}$ has this form.

[Last revised: November 14, 2014]