Math 508, Fall 2014 **Jerry L. Kazdan**

Problem Set 11

DUE: Tues. Dec.. 2, 2014. *Late papers will be accepted until 1:00 PM Wednesday.*

This week. Please read pages 154-160, 174-185 in the Rudin text and the following Notes on Convolution: `http://www.math.upenn.edu/~kazdan/508F10/convolution.pdf`

- I will have my usual office hours on Wednesday Nov. 26.
- No recitation Wednesday, Nov. 26.

1. Let X and Y be linear spaces and $L : X \to Y$ be a linear map.

   Say x_1 and x_2 are particular solutions of the equations $Lx = y_1$ and $Lx = y_2$, respectively, while $z \neq 0$ is a solution of the homogeneous equation $Lz = 0$. Answer the following in terms of x_1, x_2, and z.

   a) Find some solution of $Lx = 3y_1$.
   b) Find some solution of $Lx = -5y_2$.
   c) Find some solution of $Lx = 3y_1 - 5y_2$.
   d) Find another solution (other than z and 0) of the homogeneous equation $Lx = 0$.
   e) Find *two* solutions of $Lx = y_1$.
   f) Find another solution of $Lx = 3y_1 - 5y_2$.

2. Let $f(x) = \dfrac{\sin x}{x}$ for $x \geq 1$. Do the following improper integrals exist – and why?

$$a).\quad \int_0^\infty f(x)\, dx \qquad\qquad b).\quad \int_0^\infty |f(x)|\, dx$$

3. The *Gamma function* is defined by $\Gamma(x) := \displaystyle\int_0^\infty e^{-t} t^{x-1}\, dt$.

   a) For which real x does this improper integral converge?
   b) Show that $\Gamma(x+1) = x\Gamma(x)$ and deduce that $\Gamma(n+1) = n!$ for any integer $n \geq 0$.

4. Let $f \in C([0,\infty))$ and assume that $f(x) \geq 0$ for all $x \geq 0$. If the improper integral $\displaystyle\int_0^\infty f(x)\, dx$ exists, does this imply that f must be bounded, that is, for some constant M, we have $0 \leq f(x) < M$ for all $x \geq 0$? Give a proof or find a counterexample.

5. Let $K(x,y)$ be a continuous function of x and y for x and y in the interval $[0,1]$ and let

$$h(x) = \int_0^1 K(x,y)\,dy.$$

Show that $h(x)$ is a continuous function of x for $x \in [0,1]$.

6. In class, we showed that iff $f(x)$ and $K(x,y)$ are continuous function of x and y for x and y in the interval $0 \le x \le 1$, AND if $|\lambda|$ is sufficiently small, then the integral equation

$$u(x) = f(x) + \lambda \int_0^1 K(x,y)u(y)\,dy$$

has a unique solution $u \in C([0,1])$.

By looking at the special case where $K(x,y) \equiv 1$, and $f(x) \equiv 2$, show that the assumption that λ is sufficiently small cannot be completely eliminated.

7. In class we showed that if $A(t)$ is a square matrix and $f(t)$ a vector with both continuous for $|t| \le a$, then there is some b, $0 < b \le a$ so that the initial value problem

$$\frac{dx}{dt} = Ax, \qquad \text{with} \qquad x(0) = x_0$$

has a unique solution. Show that one can sharpen this to allowing $b = a$. One approach is to use the device that to find a fixed point of a map T, it is often enough to show that some power of T, say T^k is contracting.

8. Let $\varphi_n(t)$ be a sequence of smooth real-valued functions with the properties

$$(a)\ \varphi_n(t) \ge 0, \qquad (b)\ \varphi_n(t) = 0 \ \text{ for } \ |t| \ge 1/n, \qquad (c)\ \int_{\pi}^{\pi} \varphi_n(t)\,dt = 1.$$

Note: because of (b), this integral is only over $-1/n \le t \le 1/n$.

Extend φ to all of $\mathbb{R}$ so that it is periodic with period 2π.

Assume $f(x) \in C(\mathbb{R})$ and periodic with period 2π. Define

$$f_n(x) := \int_{-\pi}^{\pi} f(x-t)\varphi_n(t)\,dt. \tag{1}$$

Show that $f_n(x)$ is 2π periodic and converges uniformly to $f(x)$ for all $x \in [-\pi,\pi]$.

[Suggestion: Use $f(x) = f(x)\left(\int_{-\pi}^{\pi} \varphi_n(t)\,dt\right) = \int_{-\pi}^{\pi} f(x)\varphi_n(t)\,dt$. Also, note *explicitly* where you use the uniform continuity of f].

Remark: For 2π periodic continuous functions f and g, we can define $f * g$ by the rule

$$(f * g)(x) = \int_{-\pi}^{\pi} f(x-t)g(t)\,dt,$$

By a simple change of variable one sees that $f * g = g * f$. Since in equation (1) the φ_n are smooth, one can show that the approximations f_n are also smooth (see the Theorem on page 1 of the Convolution notes mentioned at the top of this assignment). Thus, this proves that you can approximate a continuous 2π periodic function *uniformly* by a smooth 2π periodic function.

Bonus Problems

[Please give this directly to Professor Kazdan]

B-1 In the the Integral Equation Example on page 75 of the notes

`http://www.math.upenn.edu/~kazdan/508F14/Notes/K-F-contracting-maps.pdf`

they prove the existence of a solution of this integral equation by showing some power of a map A is contracting. Give an alternate proof of the same result by showing that if one uses the modified uniform norm for $u \in C([a,b])$:

$$\|u\|_* = \max_{[a,b]} |u(x)e^{-\gamma x}|.$$

a) If $\|u\|$ is the usual uniform norm on $[a,b]$, show there are constants C_1 and C_2 so that for any bounded function u

$$C_1\|u\| \le \|u\|_* \le C_2\|u\|.$$

Thus both of these norms have the same convergent sequences.

b) Show that by choosing γ appropriately(depending on λ, (b-a) and M) that A itself is contracting and hence has a fixed point.

B-2 If $\varphi\colon \mathbb{R} \to \mathbb{R}$ is continuous with period P, so $\varphi(x+P) = \varphi(x)$ for all real x. Show that

$$\lim_{\lambda\to\infty} \int_0^1 f(x)\varphi(\lambda x)\,dx = \overline{\varphi} \int_0^1 f(x)\,dx,$$

where $\overline{\varphi} := \frac{1}{P}\int_0^P \varphi(t)\,dt$ is the average of φ over one period. [This generalizes both HW9 #10 and HW10 #3.]

[Last revised: September 11, 2015]