Math 508, Fall 2014 **Jerry L. Kazdan**

Problem Set 1

DUE: Tues. Sept. 9, 2014. *Late papers will be accepted until 1:00 PM Wednesday.*

LaTeX If you will be writing many documents that contain equations, it is wise to learn (and use) LaTeX. It is available on Windows, Macs, and Linux – and is FREE. See `http://www.math.upenn.edu/~kazdan/202F13/tex/`. For some students, this might be the most useful item you learn in this course.

This week. Please read all of Chapter 1 and Sections 2.1-2.14 in Chapter 2. of the Rudin text.

1. (Rudin, p. 21 #1) If $r \neq 0$ is rational and x is irrational, prove that $r + x$ and rx are irrational.

2. (Rudin, p. 22 #2) Prove that there is no rational number whose square is 12.

3. (Rudin, p. 22 #3) Prove Proposition 1.15, i.e., show that the axioms of multiplication imply the following statements:
   a) If $x \neq 0$ and $xy = xz$ then $y = z$.
   b) If $x \neq 0$ and $xy = x$ then $y = 1$.
   c) If $x \neq 0$ and $xy = 1$, then $y = 1/x$.
   d) If $x \neq 0$ then $1/(1/x) = x$.

4. (Rudin, p. 23 #17) For $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ with the Euclidean distance, prove that

$$|\mathbf{x}+\mathbf{y}|^2 + |\mathbf{x}-\mathbf{y}|^2 = 2|\mathbf{x}|^2 + 2|\mathbf{y}|^2$$

   and interpret this geometrically as a statement about a parallelogram.

   HINT: Use

$$|\mathbf{x}+\mathbf{y}|^2 = (\mathbf{x}+\mathbf{y})\cdot(\mathbf{x}+\mathbf{y}) = \mathbf{x}\cdot\mathbf{x} + 2\mathbf{x}\cdot\mathbf{y} + \mathbf{y}\cdot\mathbf{y}.$$

5. (Rudin, p. 23 #19) Suppose $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{x}$ are points in $\mathbb{R}^k$. Show that

$$|\mathbf{x}-\mathbf{a}| = 2|\mathbf{x}-\mathbf{b}|$$

   is satisfied if and only if $\mathbf{x}$ lies on some sphere $|\mathbf{x}-\mathbf{c}| = r$. [You should find $\mathbf{c} \in \mathbb{R}^k$ and $r > 0$ in terms of $\mathbf{a}$ and $\mathbf{b}$.].

[Hint: See the Hint for Problem 4 and *complete the square*:

$$\langle \mathbf{x}, \mathbf{x}\rangle - 2\langle \mathbf{c}, \mathbf{x}\rangle + \alpha = |\mathbf{x} - \mathbf{c}|^2 + (\alpha - |\mathbf{c}|^2).]$$

Generalization: For real $\lambda > 0$, $\lambda \neq 1$, consider the points $\mathbf{x} \in \mathbb{R}^k$ that satisfy

$$|\mathbf{x} - \mathbf{a}| = \lambda|\mathbf{x} - \mathbf{b}|$$

Show that these points lie on a sphere. (Thus find the center and radius of this sphere in terms of $\mathbf{a}$, $\mathbf{b}$ and λ. [You can reduce the case $\lambda > 1$ to the case $0 < \lambda < 1$ by letting $\lambda = 1/\mu$].

What if $\lambda = 1$?

6. Let S_1, S_2, S_3, ... be finite sets and $A = \cup_{j=1}^{\infty} S_j$. Show that A is countable (or finite). In other words, a countable union of finite sets is countable (or finite). [A is finite, for instance, if all the sets S_j are subsets of, say, S_1].

7. (Rudin, p.43 #2) A complex number is *algebraic* if it is a root of a polynomial $a_0 z^n + \cdots + a_n$ whose coefficients are all integers. Prove that the set of all algebraic numbers is countable. [Hint: For every positive integer N there are only finitely many equations with $n + |a_0| + \cdots + |a_n| = N$.]

8. (Rudin, p. 22 #6) The point of this problem is, for any real $b > 1$ and any real x to define b^x. So far we can only compute b^x for the special cases $x = n$ and for $x = 1/n$, where $n = 1, 2, \ldots$. First we extend this to rational x and then to all real x.

   Fix $b > 1$. Let m, n, p, q be integers with $n > 0$, $q > 0$. Define $b^{p/q} = (b^{1/q}))^p$. This equals $(b^p)^{1/q}$ since

$$[b^{p/q}]^q = [(b^{1/q})^p]^q = (b^{1/q})^{pq} = (b^{1/q})^{qp} = b^p.$$

   Now take the q^{th} root of both sides.

   Set $r = m/n = p/q$.

   a) Prove that $(b^m)^{1/n} = (b^p)^{1/q}$. Thus, it makes sense to define $b^r = (b^m)^{1/n}$.
   Solution: $b^{p/q} = b^{p(m/(np)} = b^{m/n}$

   b) If x and y are rational, prove that $b^{x+y} = b^x b^y$.
   Solution:: Note that for any integers n and k and any real $c > 0$ then $c^{n+k} = c^n c^k$. Say $x = p/q$ and $y = r/s$. Then

$$b^{x+y} = b^{(ps+qr)/(qs)} = (b^{1/qs})^{ps}(b^{1/qs})^{qr} = b^{p/q}b^{r/s} = b^x b^y.$$

   If $b > 0$, we define $b^{-x} = 1/b^x$

c) If x is real, define $B(x)$ to be the set of all numbers b^t, where t is rational and $t \le x$. Prove that for r rational

$$b^r = \sup B(r).$$

Hence it makes sense to *define* $b^x = \sup B(x)$ for all real x.

d) With this definition, prove that for all real x, y: $b^{x+y} = b^x b^y$.

9. (Rudin, P.43 #3) Prove there exist real numbers that are not algebraic.

10. (Rudin, P.43 #4) Is the set of all irrational real numbers countable? Why?

[Last revised: September 8, 2014]