Math 508, Fall 2014 **Jerry L. Kazdan**

Problem Set 4

DUE: Thurs. Sept. 25, 2014. *Late papers will be accepted until 1:00 PM Friday.*

This week. Please read pp. 47 – 63 of Chapter 3 of the Rudin text.

1. a) Calculate $\lim_{n\to\infty} \dfrac{5n+17}{n+2}$. b) If $b_n := \dfrac{3n^2-2n+17}{n^2+21n+2}$, calculate $\lim_{n\to\infty} b_n$.

2. [Rudin, p. 78 #2] Calculate $\lim_{n\to\infty} \sqrt{n^2+n}-n$.

3. Let $\{a_n > 0\}$ be a sequence of real numbers with the property that they converge to a real number $A > 0$. Prove there is a real number $c > 0$ such that $a_n > c$ for all $n = 1, 2, ...$

4. If $c > 0$ and $a_n = \dfrac{c^n}{n!}$, show that $a_n \to 0$.

5. Let $\{a_n\}$ and $\{b_n\}$ be any bounded real sequences.
   a) Show that
   $$\limsup_{n\to\infty}(a_n+b_n) \le \limsup_{n\to\infty}\, a_n + \limsup_{n\to\infty}\, b_n.$$
   b) Give an explicit example where strict inequality can occur.

6. Prove the following **Squeeze Theorem**: *Suppose $\{r_n\}$, $\{s_n\}$, and $\{t_n\}$ are real sequences such that $r_n \le s_n \le t_n$ for each n. If $\lim_{n\to\infty} r_n = \lim_{n\to\infty} t_n$ (in particular, both limits exist), then $\lim_{n\to\infty} s_n$ exists and*
   $$\lim_{n\to\infty} r_n = \lim_{n\to\infty} s_n = \lim_{n\to\infty} t_n.$$

7. Let A be the set of natural numbers which are not multiples of three. For each natural number n, let
   $$s_n := \frac{\#(A \cap \{1, \dots, n\})}{n},$$
   (i.e., s_n is the fraction of numbers from 1 to n which are not multiples of three). Show that $\lim_{n\to\infty} s_n$ exists and give its value.

   Repeat this for all the natural numbers B which are not multiples of two *or* three.

8. This question is set in $\mathbb{R}$. Let $A > 0$ and let $x_0 > 0$ be arbitrary. Define a sequence x_n recursively by

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{A}{x_n}\right), \quad n = 0,\, 1,\, 2, \ldots.$$

Show that $x_n \to \sqrt{A}$. [This is *Newton's method* for solving $x^2 - A = 0$. It converges very rapidly, roughly double the number of decimal points accuracy at each step.]

9. Is it true that every *bounded* sequence $\{a_n\}, n = 1, 2, \ldots$ of real numbers such that $|a_n - a_{n-1}| < 1/n$ for all $n \geq 2$ is convergent? Prove or give a counterexample

10. Given a real sequence $\{a_k\}$, let $C_n = \dfrac{a_1 + \cdots + a_n}{n}$ be the sequence of averages (*arithmetic mean*).

    a) Give an example where the a_n's doesn't converge but the averages do converge.

    b) If a_k converges to A, show that also C_n converges, and to A.

    c) If the $a_k \geq 0$ and the averages converge, must the a_k's be bounded? (Proof or counterexample)

[Last revised: September 18, 2014]