Problem Set 5

DUE: Thurs. Oct. 2, 2014. *Late papers will be accepted until 1:00 PM Friday.*

This week. Please read all of Chapter 3 of the Rudin text. Although there are 12 problems, many of them should be quite routine.

1. (Rudin, p.78 #6d) Investigate the convergence or divergence of $\sum_n \frac{1}{1+z^n}$ (complex z).

2. (Rudin p. 79 #8) Assume $a_n > 0$. If $\sum a_n$ converges and $\{b_n\}$ is bounded, prove that $\sum a_n b_n$ converges.

3. (Rudin p. 79 #9) Find the radius of convergence of each of the following power series.

   a). $\sum n^3 z^n$, b). $\sum \frac{2^n}{n!} z^n$ c). $\sum n! z^n$

4. Determine the limit of the sequence $\{a_n\}$ defined by $a_0 := 0$ and $a_n := a_{n-1}^2 + \frac{4}{25}$ for each $n \geq 1$.

5. (Rudin, p. 78 #7) If $a_n \geq 0$ and $\displaystyle\sum_{n=1}^{\infty} a_n$ converges, show that $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{a_n}}{n}$ converges. [SUGGESTION: Use $2|xy| \leq x^2 + y^2$ for all real x, y.]

6. Find a number N so that $1 + \dfrac{1}{2} + \dfrac{1}{3} + \cdots + \dfrac{1}{N} > 100$.

7. Determine if the series $1 + \dfrac{1}{2} - \dfrac{1}{3} - \dfrac{1}{4} + \dfrac{1}{5} + \dfrac{1}{6} - \dfrac{1}{7} - \dfrac{1}{8} + \cdots$ converges or diverges (the sign pattern is $+ + - - + + - - + + - - + + \ldots$).

8. In R^n, for a vector $x = (x_1, x_2, \ldots, x_n)$, for any $1 \leq p < \infty$ we can define the norm

$$\|x\|_p := \left[\, |x_1|^p + |x_2|^p + \cdots + |x_n|^p \,\right]^{1/p} = \left[\sum_{j=1}^{n} |x_j|^p \right]^{1/p}$$

   (we'll prove the triangle inequality later).

   a) Show that $\displaystyle\max_{1 \leq j \leq n} \{|x_j|\} \leq \|x\|_p \leq n^{1/p} \max_{1 \leq j \leq n} \{|x_j|\}$.

   b) Show that $\displaystyle\lim_{p \to \infty} \|x\|_p = \max_{1 \leq j \leq n} \{|x_j|, \}$. [Thus we define $\displaystyle\|x\|_\infty = \max_{1 \leq j \leq n} \{|x_j|, \}$.]

9. In http://www.math.upenn.edu/~kazdan/508F15/Notes/Basic-examples.pdf we defined the space ℓ_2. Show that it is complete. You can see a similar proof for ℓ_1 in http://www.math.upenn.edu/~kazdan/508F10/completeness-l_1-2010.pdf

10. Let A be an $n \times n$ real or complex matrix. We use the norm of Homework Set 3 #6.
    a) Compute $(I - A)(I + A + A^2 + A^3 + A^4 + \cdots + A^N)$.
    b) If $|A| < 1$, show that the matrix $I - A$ is invertible. $The first step is to show that$ $S_k := I + A + A^2 + \cdots + A_k$ is a Cauchy sequence, so there is some matrix S to which it converges. Then use Part a) to show that $(I - A)Q = I$.]
    c) Show that the set of invertible $n \times n$ (real or complex) matrices is open. Thus if a matrix is invertible, so are all nearby matrices.

11. Show that the set $O(n)$ of all $n \times n$ real orthogonal matrices is compact. Note $O(n) \subset \mathbb{R}^{n^2}$ [REMARK: There are many (equivalent) definitions for an orthogonal matrix. Use whichever you prefer.]

12. a) Let $\{a_n\}$ be a sequence of real numbers with the property that

$$|a_{k+1} - a_k| \le \tfrac{1}{2}|a_k - a_{k-1}|, \qquad k = 1, 2, \ldots .$$

Show that this sequence converges to some real number. HINT: Show that $\{a_n\}$ is a Cauchy sequence. First step: $|a_{k+1} - a_k| \le \left(\frac{1}{2}\right)^k |a_1 - a_0|$. [One can replace 1/2 by any c where $0 < c < 1$.]

    b) In a complete metric space (X, d) say you have a map $f : X \to X$ that is *contracting* in the sense that for any $p.\, q \in X$

$$d(f(p), f(q)) \le \tfrac{1}{2} d(p, q).$$

Given any starting point $p_0 \in X$ define the points $p_{k+1} = f(p_k)$, $k = 0, 1, 2, \ldots$. Show that the p_k converge to a some point $p \in X$ and that p is a *fixed point of f* in the sense that $f(p) = p$.
Show that this fixed point is unique, that is, if $f(p) = p$ and $f(q) = q$, then $p = q$.

[Last revised: September 30, 2014]