Math 508, Fall 2014 **Jerry L. Kazdan**

Problem Set 7

DUE: Thurs. Oct. 30, 2014. *Late papers will be accepted until 1:00 PM Friday.*

This week. Please read all of Chapter 5 in the Rudin text.

Note: We say a function is *smooth* if its derivatives of all orders exist and are continuous.

1. Let $f : [a, \infty) \to \mathbb{R}$ be a smooth function whose first derivative is bounded: $|f'(x)| \le M$ for all $x \ge a$. Prove that it is uniformly continuous on $[a, \infty)$.

   As immediate examples, $x^{1/3}$ is uniformly continuous for all $x \ge 1$ and $\cos x$ is uniformly continuous for all x.

2. a) Show that $\sin x$ is not a polynomial.

   b) Show that $\sin x$ is not a rational function, that is, it cannot be the quotient of two polynomials.

   c) Let $f(t)$ be periodic with period 1, so $f(t+1) = f(t)$ for all real t. If f is not a constant, show that it cannot be a rational function. that is, f cannot be the quotient of two polynomials.

   d) Show that e^x is not a rational function.

3. Show that $\lim_{n\to\infty}(n+1)^{1/7} - n^{1/7} = 0$. [HINT: Mean Value Theorem]

4. Say a smooth function $f : \mathbb{R} \to \mathbb{R}$ has the properies $f(0) = 3$, $f(1) = 2$, and $f(3) = 8$. Show there is at least one point c in the interval $0 < x < 3$ where $f''(c) > 0$; in fact, find some explicit constant $M > 0$ such that $f''(c) \ge M$.

5. a) If a smooth function $f(x)$ has the property that $f''(x) \ge 0$ for all x, show that it is *convex*, that is, at every point the graph of the curve $y = f(x)$ lies above all its tangent lines.

   b) Let $v(x)$ be a smooth real-valued function for $0 \le x \le 1$. If $v(0) = v(1) = 0$ and $v''(x) > 0$ for all $0 \le x \le 1$, show that $v(x) \le 0$ for all $0 \le x \le 1$.

   c) Prove that the function e^x is convex.

   d) Show that $e^x \ge 1 + x$ for all real x.

6. a) Let $p(x) := x^3 + cx + d$, where c, and d are real. Under what conditions on c and d does this has three distinct real roots? [ANSWER:$c < 0$ and $d^2 < -4c^3/27$].

b) Generalize to the real polynomial $p(x) := ax^3 + bx^2 + cx + d$ ($a \neq 0$) by a change of variable $t = x - \alpha$ (with a clever choice of α) to reduce to the above special case.

7. Let

$$p_n(x) := \left(\frac{d}{dx}\right)^n (1 - x^2)^n.$$

This is a polynomial of degree n. Show that it has n real distinct zeroes, all in the interval $-1 < x < 1$.

8. a) For any integer $n \geq 0$, show that $\lim_{x \searrow 0} \frac{e^{-1/x}}{x^n} = 0$.

b) Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} e^{-\frac{1}{x}} & \text{for } x > 0, \\ 0 & \text{for } x \leq 0, \end{cases}.$$

Sketch the graph of f.

c) Show that f is a smooth function for all real x.

d) Show that each of the following are smooth and sketch their graphs:

$$g(x) = f(x)f(1-x) \qquad h(x) = \frac{f(x)}{f(x) + f(1-x)}$$

$$k(x) = h(x)h(4-x) \qquad K(x) = k(x+2),$$

$$\varphi(x,y) = K(x)K(y),\ (x,y) \in \mathbb{R}^2 \qquad \Phi(x) = K(\|x\|),\ x = (x_1, x_2) \in \mathbb{R}^2$$

$$G(x) = \sum_{n=-\infty}^{\infty} g(x - n/2) \qquad p(x) = \frac{g(x)}{G(x)}$$

e) Define $p_n(x) = p(x - n/2)$. Show that $\sum_{n=-\infty}^{\infty} p_n(x) \equiv 1$. This is called a (smooth) *partition of unity* for the real line. Note that for any function $u(x)$ if we let $u_n(x) := p_n(x)u(x)$, then $u(x) = \sum_{n=-\infty}^{\infty} u_n(x)$. This gives a way to localize a function defined of the whole real line.

Bonus Problem

[Please give this directly to Professor Kazdan]

B-1 [Interpolation] Let $x_0 < x_1 < x_2$ be distinct real numbers and $f(x)$ a smooth function.

a) Show there is a unique quadratic polynomial $p(x)$ with the property that $p(x_j) = f(x_j)$ for $j = 0, 1, 2$.

b) [Remainder term in interpolation] If b is in the open interval (x_0, x_2) with $b \neq x_j$, $j = 0, 1, 2$, show there is a point c (depending on b) in the interval (x_0, x_2) so that

$$f(b) = p(b) + \frac{f'''(c)}{3!}(b - x_0)(b - x_1)(b - x_2).$$

This estimate is related to the procedure used to find the remainder in a Taylor polynomial.

[SUGGESTION: Define the constant M by

$$f(b) = p(b) + M(b - x_0)(b - x_1)(b - x_2),$$

and look at

$$g(x) := f(x) - [p(x) + M(x - x_0)(x - x_1)(x - x_2)].]$$

[Last revised: October 3, 2023]