Problem Set 7
DUE: Thurs. Oct. 30, 2014. Late papers will be accepted until 1:00 PM Friday.

This week. Please read all of Chapter 5 in the Rudin text.

Note: We say a function is smooth if its derivatives of all orders exist and are continuous.

1. Let $f : [a, \infty) \to \mathbb{R}$ be a smooth function whose first derivative is bounded: $|f'(x)| \leq M$ for all $x \geq a$. Prove that it is uniformly continuous on $[a, \infty)$. As immediate examples, $x^{1/3}$ is uniformly continuous for all $x \geq 1$ and $\cos x$ is uniformly continuous for all $x$.

2. a) Show that $\sin x$ is not a polynomial.
   b) Show that $\sin x$ is not a rational function, that is, it cannot be the quotient of two polynomials.
   c) Let $f(t)$ be periodic with period 1, so $f(t + 1) = f(t)$ for all real $t$. If $f$ is not a constant, show that it cannot be a rational function. That is, $f$ cannot be the quotient of two polynomials.
   d) Show that $e^x$ is not a rational function.

3. Show that $\lim_{n \to \infty} (n + 1)^{1/7} - n^{1/7} = 0$. [HINT: Mean Value Theorem]

4. Say a smooth function $f : \mathbb{R} \to \mathbb{R}$ has the properties $f(0) = 3$, $f(1) = 2$, and $f(3) = 8$. Show there is at least one point $c$ in the interval $0 < x < 3$ where $f''(c) > 0$; in fact, find some explicit constant $M > 0$ such that $f''(c) \geq M$.

5. a) If a smooth function $f(x)$ has the property that $f''(x) \geq 0$ for all $x$, show that it is convex, that is, at every point the graph of the curve $y = f(x)$ lies above all its tangent lines.
   b) Let $v(x)$ be a smooth real-valued function for $0 \leq x \leq 1$. If $v(0) = v(1) = 0$ and $v''(x) > 0$ for all $0 \leq x \leq 1$, show that $v(x) \leq 0$ for all $0 \leq x \leq 1$.
   c) Prove that the function $e^x$ is convex.
   d) Show that $e^x \geq 1 + x$ for all real $x$.

6. a) Let $p(x) := x^3 + cx + d$, where $c$, and $d$ are real. Under what conditions on $c$ and $d$ does this have three distinct real roots? [Answer: $c < 0$ and $d^2 < -4c^3/27$].

1
b) Generalize to the real polynomial \( p(x) := ax^3 + bx^2 + cx + d \) (\( a \neq 0 \)) by a change of variable \( t = x - \alpha \) (with a clever choice of \( \alpha \)) to reduce to the above special case.

7. Let

\[ p_n(x) := \left( \frac{d}{dx} \right)^n (1 - x^2)^n. \]

This is a polynomial of degree \( n \). Show that it has \( n \) real distinct zeroes, all in the interval \(-1 < x < 1\).

8. a) For any integer \( n \geq 0 \), show that \( \lim_{x \to 0} \frac{e^{-1/x}}{x^n} = 0 \).

b) Define \( f : \mathbb{R} \to \mathbb{R} \) by

\[ f(x) = \begin{cases} e^{-\frac{x}{2}} & \text{for } x > 0, \\ 0 & \text{for } x \leq 0. \end{cases} \]

Sketch the graph of \( f \).

c) Show that \( f \) is a smooth function for all real \( x \).

d) Show that each of the following are smooth and sketch their graphs:

\[
\begin{align*}
g(x) &= f(x) f(1 - x) \\
h(x) &= \frac{f(x)}{f(x) + f(1 - x)} \\
k(x) &= h(x) h(4 - x) \\
K(x) &= k(x + 2) \\
\varphi(x, y) &= K(x) K(y), \ (x, y) \in \mathbb{R}^2 \\
\Phi(x) &= K(\|x\|), \ x = (x_1, x_2) \in \mathbb{R}^2
\end{align*}
\]

**Bonus Problem**

[Please give this directly to Professor Kazdan]

B-1 [Interpolation] Let \( x_0 < x_1 < x_2 \) be distinct real numbers and \( f(x) \) a smooth function.

a) Show there is a unique quadratic polynomial \( p(x) \) with the property that \( p(x_j) = f(x_j) \) for \( j = 0, 1, 2 \).

b) [Remainder term in interpolation] If \( b \) is in the open interval \((x_0, x_2)\) with \( b \neq x_j, j = 0, 1, 2 \), show there is a point \( c \) (depending on \( b \)) in the interval \((x_0, x_2)\) so that

\[
f(b) = p(b) + \frac{f'''(c)}{3!} (b - x_0)(b - x_1)(b - x_2).
\]

This estimate is related to the procedure used to find the remainder in a Taylor polynomial.
Suggestion: Define the constant $M$ by

$$f(b) = p(b) + M(b - x_0)(b - x_1)(b - x_2),$$

and look at

$$g(x) := f(x) - [p(x) + M(x - x_0)(x - x_1)(x - x_2)],$$

[Last revised: October 27, 2014]