Math 508, Fall 2014 **Jerry L. Kazdan**

Problem Set 9

DUE: Thurs. Nov. 13, 2014. *Late papers will be accepted until 1:00 PM Friday.*

This week. Please read Chapter 7 pages 143-154 in the Rudin text.

Note: We say a function is *smooth* if its derivatives of all orders exist and are continuous.

1. Assume the function $f(x)$ is *even*, that is, $f(-x) = f(x)$ (Example: $\cos 3x$) and $g(x)$ is *odd*, that is, $g(-x) = -g(x)$ (Example: $\sin x \cos 3x$). Assuming that f and g are Riemann integrable, show that for any $c > 0$

$$\int_{-c}^{c} f(x)\,dx = 2\int_{0}^{c} f(x)\,dx, \qquad \int_{-c}^{c} g(x)\,dx = 0, \qquad \int_{-c}^{c} f(x)g(x)\,dx = 0.$$

   There are two approaches, one geometric the other by a computation.

2. Let $f(x)$ be a continuous function that satisfies $\displaystyle\int_0^x f(t)e^{3t}\,dt = c + x - \cos(x^2)$. Find the function f and the constant c.

3. Say you want to compute $\displaystyle\int_0^2 \sqrt{9+x^4}\,dx$ using Riemann sums with a partition

$$P = \{0 = x_0 < x_1 < \cdots < x_{N-1} < x_N = 2\}.$$

   Assume these nodes are equally spaced, so $x_j - x_{j-1} = \Delta x = (2-0)/N$. If in the intervals $x_{j-1} \le x \le x_j$ you evaluate the integrand at the left end points, how large should you pick N so that the error in your value of the integral is less than $1/100$? Justify your assertion. [Note that we are not seeking the smallest N, just one that works.]

4. If $f(s)$ is a smooth function, and c is a constant, let $\displaystyle u(x,t) = \int_{x-ct}^{x+ct} f(s)\,ds$. Show that $u(x,t)$ is a solution of the *wave equation*

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}.$$

   [You may use the standard elementary properties of partial derivatives.]

5. This problem concerns finding a solution $u(x)$ of the *boundary value problem*

$$u'' + c^2 u = f(x) \qquad \text{on} \quad [0,\, \pi] \quad \text{with} \quad u(0) = 0,\ u(\pi) = 0 \tag{1}$$

on the interval $[0,\, \pi]$. Here $c > 0$ is a constant.

a) Find a formula for the general solution of the *initial value problem*

$$u'' + c^2 u = f(x) \qquad \text{with} \quad u(0) = \alpha,\ u'(0) = \beta. \tag{2}$$

[The standard approach use the method *variation of parameters* discussed, for instance, in Math 240 texts and in a Google search.]

b) If $0 < c < 1$ use the formula you found to find a solution of the problem (1). Is this the unique solution?

c) If there is a solution of (1) for the case $c = 1$, so $u'' + u = f$, show that f must satisfy

$$\int_0^\pi f(x) \sin x \, dx = 0. \tag{3}$$

d) Use the result of part a). to show that for $c = 1$ if f satisfies condition (3), then the problem (1) *has* a solution. Thus for $c = 1$ the orthogonality condition (3) is necessary and sufficient for a solution of (1) to exist. Is this solution unique?

6. Define the differential operator L by $Lw = -w'' + c(x)w$ on the interval $J = [a,\, b]$, where $c(x)$ is some continuous function. Define the inner product by

$$\langle f,\, g\rangle = \int_a^b f(x) g(x) \, dx.$$

a) If both u and v are zero on the boundary of J, show that

$$\langle Lu,\, v\rangle = \langle u,\, Lv\rangle.$$

b) Say $Lu = \lambda_1 u$ and $Lv = \lambda_2 v$, where both u and v are zero on the boundary of J. If $\lambda_1 \neq \lambda_2$, show that $u(x)$ and $v(x)$ are orthogonal, that is, $\langle u,\, v\rangle = 0$, [A special case is $Lw = w''$ on the interval $[0,\, \pi]$, $u(x) = \sin x$, $v(x) = \sin 2x$.]

c) Say $Lu = 0$ and $Lv = f(x)$, where both u and v are zero on the boundary of J. Show that f must be orthogonal to u, that is, $\langle u,\, f\rangle = 0$. Thus, if a solution exists, then f must satisfy this. [A special case is Problem 4c).]

7. Compute the arc length of the following helix in $\mathbb{R}^3$:

$$X(t) = (\cos t,\, \sin t,\, t) \quad \text{for} \quad 0 \le t \le 4\pi.$$

8. Let $X(t)$ be the curve $X(t) = (t,\, t\sin(1/t))$ for $0 < t \le 2/\pi$, while $X(0) = 0$. Show that this curve is *not* rectifiable: it has infinite arc length.

9. Let $f : [0, 1] \to \mathbb{R}$ be a continuous function.
   a) If $\int_0^1 f(x)\,dx = 0$, prove that $f(x)$ is positive somewhere and negative somewhere in this interval (unless it is identically zero).
   b) Use this to show that $\|f\|_1 := \displaystyle\int_0^1 |f(x)|\,dx$ is a norm on $C([0,1])$.
   c) Show that $C([0,1])$ with this norm is *not* complete.

10. Compute $\displaystyle\lim_{\lambda\to\infty} \int_0^1 |\sin(\lambda x)|\,dx$.

11. Let $f \in C([0,\infty))$ be a continuous function with the property that $\lim_{x\to\infty} f(x) = c$. Show that

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T f(x)\,dx = c.$$

12. a) If $f : [0, 1] \to \mathbb{R}$ is a continuous function with the property that $\int_0^1 f(x)g(x)\,dx = 0$ for *all* continuous functions g, prove that $f(x) = 0$ for all $x \in [0, 1]$.
    b) If $f : [0, 1] \to \mathbb{R}$ is a continuous function with the property that $\int_0^1 f(x)g(x)\,dx = 0$ for *all* C^1 functions g that satisfy $g(0) = g(1) = 0$, must it be true that $f(x) = 0$ for all $x \in [0, 1]$? Proof or counterexample.

Bonus Problems

[Please give this directly to Professor Kazdan]

B-1 Let $f(x)$ be a continuous function for $0 \le x \le 1$. Evaluate $\displaystyle\lim_{n\to\infty}\int_0^1 nf(x)x^n\,dx$. (Justify your assertions.)

B-2 For $x > 0$ define the function

$$H(x) = \int_1^x \frac{1}{t}\,dt.$$

Since the integrand, $1/t$ is a continuous function for $t > 0$, this is Riemann integrable.

Use the definition of the Riemann integral directly to show that for any $y > 0$,

$$H(x) + H(y) = H(xy), \tag{4}$$

thus establishing that $H(x)$ has the basic property of the logarithm.
Suggestion: First prove (4) assuming $x \ge 1$ (and any $y > 0$). If $0 < x \le 1$, make a change of variable to reduce to the case $x \ge 1$.

[Last revised: November 16, 2014]