Problem Set 9
Due: Thurs. Nov. 13, 2014. Late papers will be accepted until 1:00 PM Friday.

This week. Please read Chapter 7 pages 143-154 in the Rudin text.

Note: We say a function is smooth if its derivatives of all orders exist and are continuous.

1. Assume the function $f(x)$ is even, that is, $f(-x) = f(x)$ (Example: $\cos 3x$) and $g(x)$ is odd, that is, $g(-x) = -g(x)$ (Example: $\sin x \cos 3x$). Assuming that $f$ and $g$ are Riemann integrable, show that for any $c > 0$

$$
\int_{-c}^{c} f(x) \, dx = 2 \int_{0}^{c} f(x) \, dx, \quad \int_{-c}^{c} g(x) \, dx = 0, \quad \int_{-c}^{c} f(x)g(x) \, dx = 0.
$$

There are two approaches, one geometric the other by a computation.

2. Let $f(x)$ be a continuous function that satisfies $\int_{0}^{x} f(t)e^{3t} \, dt = c + x - \cos(x^2)$.

Find the function $f$ and the constant $c$.

3. Say you want to compute $\int_{0}^{2} \sqrt{9 + x^4} \, dx$ using Riemann sums with a partition

$$
P = \{0 = x_0 < x_1 < \cdots < x_{N-1} < x_N = 2\}.
$$

Assume these nodes are equally spaced, so $x_j - x_{j-1} = \Delta x = (2 - 0)/N$. If in the intervals $x_{j-1} \leq x \leq x_j$ you evaluate the integrand at the left end points, how large should you pick $N$ so that the error in your value of the integral is less than 1/100? Justify your assertion. [Note that we are not seeking the smallest $N$, just one that works.]

4. If $f(s)$ is a smooth function, and $c$ is a constant, let $u(x,t) = \int_{x-ct}^{x+ct} f(s) \, ds$.

Show that $u(x,t)$ is a solution of the wave equation

$$
\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}.
$$

[You may use the standard elementary properties of partial derivatives.]
5. This problem concerns finding a solution $u(x)$ of the boundary value problem

$$u'' + c^2 u = f(x) \quad \text{on} \quad [0, \pi] \quad \text{with} \quad u(0) = 0, \ u(\pi) = 0 \quad (1)$$

on the interval $[0, \pi]$. Here $c > 0$ is a constant.

a) Find a formula for the general solution of the initial value problem

$$u'' + c^2 u = f(x) \quad \text{with} \quad u(0) = \alpha, \ u'(0) = \beta. \quad (2)$$

[The standard approach use the method variation of parameters discussed, for instance, in Math 240 texts and in a Google search.]

b) If $0 < c < 1$ use the formula you found to find a solution of the problem (1). Is this the unique solution?

c) If there is a solution of (1) for the case $c = 1$, so $u'' + u = f$, show that $f$ must satisfy

$$\int_0^\pi f(x) \sin x \, dx = 0. \quad (3)$$

d) Use the result of part a) to show that for $c = 1$ if $f$ satisfies condition (3), then the problem (1) has a solution. Thus for $c = 1$ the orthogonality condition (3) is necessary and sufficient for a solution of (1) to exist. Is this solution unique?

6. Define the differential operator $L$ by $Lw = -w'' + c(x) w$ on the interval $J = [a, b]$, where $c(x)$ is some continuous function. Define the inner product by

$$\langle f, g \rangle = \int_a^b f(x) g(x) \, dx.$$

a) If both $u$ and $v$ are zero on the boundary of $J$, show that

$$\langle Lu, v \rangle = \langle u, Lv \rangle.$$

b) Say $Lu = \lambda_1 u$ and $Lv = \lambda_2 v$, where both $u$ and $v$ are zero on the boundary of $J$. If $\lambda_1 \neq \lambda_2$, show that $u(x)$ and $v(x)$ are orthogonal, that is, $\langle u, v \rangle = 0$. [A special case is $Lw = w''$ on the interval $[0, \pi]$, $u(x) = \sin x$, $v(x) = \sin 2x$.]

c) Say $Lu = 0$ and $Lv = f(x)$, where both $u$ and $v$ are zero on the boundary of $J$. Show that $f$ must be orthogonal to $u$, that is, $\langle u, f \rangle = 0$. Thus, if a solution exists, then $f$ must satisfy this. [A special case is Problem 4c.]

7. Compute the arc length of the following helix in $\mathbb{R}^3$:

$$X(t) = (\cos t, \sin t, t) \quad \text{for} \quad 0 \leq t \leq 4\pi.$$

8. Let $X(t)$ be the curve $X(t) = (t, t \sin(1/t))$ for $0 < t \leq 2/\pi$, while $X(0) = 0$. Show that this curve is not rectifiable: it has infinite arc length.
9. Let $f : [0, 1] \rightarrow \mathbb{R}$ be a continuous function.
   a) If $\int_0^1 f(x) \, dx = 0$, prove that $f(x)$ is positive somewhere and negative somewhere in this interval (unless it is identically zero).
   b) Use this to show that $\|f\|_1 := \int_0^1 |f(x)| \, dx$ is a norm on $C([0, 1])$.
   c) Show that $C([0, 1])$ with this norm is not complete.

10. Compute $\lim_{\lambda \to \infty} \int_0^1 |\sin(\lambda x)| \, dx$.

11. Let $f \in C([0, \infty))$ be a continuous function with the property that $\lim_{x \to \infty} f(x) = c$. Show that
    $$\lim_{T \to \infty} \frac{1}{T} \int_0^T f(x) \, dx = c.$$ 

12. a) If $f : [0, 1] \rightarrow \mathbb{R}$ is a continuous function with the property that $\int_0^1 f(x)g(x) \, dx = 0$ for all continuous functions $g$, prove that $f(x) = 0$ for all $x \in [0, 1]$.
   b) If $f : [0, 1] \rightarrow \mathbb{R}$ is a continuous function with the property that $\int_0^1 f(x)g(x) \, dx = 0$ for all $C^1$ functions $g$ that satisfy $g(0) = g(1) = 0$, must it be true that $f(x) = 0$ for all $x \in [0, 1]$? Proof or counterexample.

**Bonus Problems**

[Please give this directly to Professor Kazdan]

B-1 Let $f(x)$ be a continuous function for $0 \leq x \leq 1$. Evaluate $\lim_{n \to \infty} \int_0^1 nf(x)x^n \, dx$. (Justify your assertions.)

B-2 For $x > 0$ define the function
$$H(x) = \int_1^x \frac{1}{t} \, dt.$$ 

Since the integrand, $1/t$ is a continuous function for $t > 0$, this is Riemann integrable. Use the definition of the Riemann integral directly to show that for any $y > 0$,
$$H(x) + H(y) = H(xy),$$ 

thus establishing that $H(x)$ has the basic property of the logarithm. 

Suggestion: First prove (4) assuming $x \geq 1$ (and any $y > 0$). If $0 < x \leq 1$, make a change of variable to reduce to the case $x \geq 1$.

[Last revised: November 16, 2014]