Partition of Unity

Note: We say a function is *smooth* if its derivatives of all orders exist and are continuous.

1. a) For any integer $n \geq 0$, show that $\displaystyle\lim_{x \searrow 0} \frac{e^{-1/x}}{x^n} = 0$.

   b) Define $f : \mathbb{R} \to \mathbb{R}$ by
   $$f(x) = \begin{cases} e^{-\frac{1}{x}} & \text{for } x > 0, \\ 0 & \text{for } x \leq 0, \end{cases}.$$
   Sketch the graph of f.

   c) Show that f is a smooth function for all real x.

   d) Sketch the graphs of the following:
   $$g(x) = f(x)f(1-x) \qquad h(x) = \frac{f(x)}{f(x) + f(1-x)}$$
   $$k(x) = h(x)h(4-x) \qquad K(x) = k(x+2),$$
   $$\varphi(x, y) = K(x)K(y),\ (x, y) \in \mathbb{R}^2 \qquad \Phi(x) = K(\|x\|),\ x = (x_1, x_2) \in \mathbb{R}^2$$

   e) Let $G(x) = \sum_{n=-\infty}^{\infty} g(x - n/2)$. Sketch $p(x) = \frac{g(x)}{G(x)}$.

   f) Define $p_n(x) = p(x - n/2)$. Show that $\sum_{n=-\infty}^{\infty} p_n(x) \equiv 1$. This is called a (smooth) *partition of unity* for the real line. Note that for any function $u(x)$ if we let $u_n(x) := p_n(x)u(x)$, then $u(x) = \sum_{n=-\infty}^{\infty} u_n(x)$. This gives a way to localize a function defined of the whole real line.